GEOMETRIC PROGRESSION

A sequence of non-zero numbers is called a geometric progression (abbreviated as G.P.) if the ratio of a term and the term preceding to it is always a constant quantity.

The constant ratio is called the common ratio of the G.P.

In other words, a sequence , $a_1, a_2, a_3, \ldots, a_n, \ldots$ is called a geometric progression if

$$\frac{a_{n+1}}{a_n} = \text{constant for all } n \in N.$$

Geometric series

If $a_1, a_2, a_3, \ldots a_n, \ldots$ is a G.P. , then the expression

$a_1 + a_2 + a_3 + \ldots + a_n + \ldots$ is called a geometric series.

Note that the geometric series is finite or infinite according as the corresponding G.P. consists of finite or infinite number of terms.

The nth or General term of a G.P.

THEOREM 1: **Prove that the nth term of a G.P. with first term a and common ratio r is given by** $\mathbf{a_n = ar^{n-1}}$

PROOF: Let $a_1, a_2, a_3, \ldots, a_n, \ldots$ be the given G.P. Then,

$a_1 = a \Rightarrow a_1 = ar^{1-1}$

Since $a_1, a_2, a_3, \ldots, a_n, \ldots$ is a G.P. with common ratio r. Therefore,

$$\frac{a_2}{a_1} = r \Rightarrow a_2 = a_1 r \Rightarrow a_2 = ar \Rightarrow a_2 = ar^{2-1}$$

$$\frac{a_3}{a_2} = r \Rightarrow a_3 = a_2 r \Rightarrow a_3 = (ar)\, r \Rightarrow a_3 = ar^2 \Rightarrow a_3 = ar^{3-1}.$$

$$\frac{a_4}{a_3} = r \Rightarrow a_4 = a_3 r \Rightarrow a_4 = (ar^2)\, r \Rightarrow a_4 = ar^3 \Rightarrow a_4 = ar^{4-1}$$

Continuing in this manner, we get

$$a_n = ar^{n-1}$$

NOTE : It follows from the above discussion that if a is the first term and r is the common ratio of a G.P. , then the G.P. can be written as $a, ar, ar^2, \ldots, ar^{n-1}$ or, $a, ar, ar^2, ar^3, ar^4, \ldots, ar^{n-1}, \ldots$ according as it is finite or infinite.

nth Term from the end of a finite G.P.

Theorem 2: **Prove that the nth term from the end of a finite G.P. consisting of m terms is $\mathbf{ar^{m-n}}$, where a is the first term and r is the common ratio of the G.P.**

Proof: Since the G.P. consists of m terms. Therefore,

nth term from the end = (m – n + 1)th term from the beginning

$= ar^{m-n}$

Theorem 3: **Prove that the nth term from the end of a G.P. with last term ℓ and common ratio r is given by $a_n = \ell\left(\frac{1}{r}\right)^{n-1}$**

Proof : Clearly, when we look at the terms of a G.P. from the last term and move towards the beginning we find that the progression is a G.P. with common ratio 1/r, So, nth term from the end = $\ell\left(\frac{1}{r}\right)^{n-1}$

Illustration 40

Find the 10th term of the sequence 1, 2, 4, 8 , ……

The sequence 1, 2, 4, 8, …. is a G.P. with C.R. = 2.

nth term in G.P. = ar^{n-1}.

Here a = 1^{st} term = 1, r = C.R. = 2 and n = 10

$\therefore$ 10th term = $1.\,(2)^{10-1} = 2^9 = 2^4 \times 2^5 = 16 \times 32 = 512$.

Illustration 41

(i) Write down the 20th term of the G.P. 1, –1, 1, –1, …..

(ii) Write down the 5th term of the series $\frac{1}{4} - \frac{1}{2} + 1 - ....$

(i) Here a = 1^{st} term = 1 and r = Common ratio = –1

$\therefore$ 20th term = $a.\,(r)^{20-1} = 1\,.\,(-1)^{19} = 1.\,(-1) = -1$

(ii) The given series is a G.P. with C.R. = –2 and a = $\frac{1}{4}$

$\therefore$ 5th term = $a(r)^{5-1} = \frac{1}{4} \times 16 = 4$

Illustration 42

In a G.P. the product of the first three terms is 27/8. Find the middle term. (Assume that the terms are positive).

Let a be the first term and r be the common ratio of the G.P. Then the first three terms of the G.P. are a, ar, ar^2.

$\therefore$ $a \times ar \times ar^2 = \frac{27}{8}$, or, $a^3r^3 = \frac{27}{8}$ or, $(ar)^3 = \left(\frac{3}{2}\right)^3$; $\therefore$ $ar = \frac{3}{2}$

Hence the middle term is ar = $\frac{3}{2}$

Illustration 43

The 4th, 7th and 10th terms of a G.P. are a, b, c respectively. Show that $b^2 = ac$.

Let a_1 be the first term and r the common ratio of the G.P.

Then 4^{th} term = $a_1(r)^{4-1} = a_1r^3$, 7th term = $a_1\,(r)^{7-1} = a_1r^6$

and 10th term = $a_1(r)^{10-1} = a_1r^9$.

$\therefore$ $a = a_1r^3$, $b = a_1r^6$ and $c = a_1r^9$.

$\therefore$ $b^2 = (a_1r^6)^2 = a_1^2r^{12}$ and $ac = a_1r^3\,.\,a_1r^9 = a_1^2\,r^{12}$.

Hence $b^2 = ac$.

Illustration 44

The 6th and the 13th terms of a G.P. are 64 and 8192 respectively. Find the 10th and nth terms of the G.P.

Let a be the first term and r be the common ratio of the G.P. Then, nth term of the G.P. = ar^{n-1}.

$\therefore$ 6th term = $ar^{6-1} = ar^5$, and 13th term = $ar^{13-1} = ar^{12}$.

$\therefore$ $ar^5 = 64$ .....(1) and $ar^{12} = 8192$ .....(2)

Dividing (2) by (1), $\frac{ar^{12}}{ar^5} = \frac{8192}{64}$, or $r^7 = 128 = 2^7$; $\therefore$ r = 2

From (1), $a.2^5 = 64$, or, 32a = 64 ; $\therefore$ a = 2

10th term of the G.P. = $ar^{10-1} = 2.(2)^9 = 2^{10} = 1024$

and nth term of the G.P. = $ar^{n-1} = 2.(2)^{n-1} = 2^n$.

Illustration 45

Is it possible that three different numbers a, b, c may be both in A.P. and in G.P. Give reasons for your answer.

If possible, let a, b, c be both in A.P. and in G.P. Then, we have

a + c = 2b .....(1) and $ac = b^2$ ....(2)

From (1), $(a + c)^2 = 4b^2 = 4ac$.

or, $(a + c)^2 - 4ac = 0$, or, $(a - c)^2 = 0$; $\therefore$ a = c

But a, b, c are three different numbers (given) $\therefore$ a $\neq$ c.

Hence three different numbers a, b, c cannot be both in A.P. and in G.P.

Illustration 46

Find the 5th term of the progression

$$\mathbf{1, \frac{(\sqrt{2}-1)}{2\sqrt{3}}, \left(\frac{3-2\sqrt{2}}{12}\right), \left(\frac{5\sqrt{2}-7}{24\sqrt{3}}\right)}$$

Clearly, the given progression is a G.P. with first term a = 1 and common ratio r = $\left(\frac{\sqrt{2}-1}{2\sqrt{3}}\right)$. So, its 5^{th} term is given by

$$a_5 = ar^{(5-1)} = 1 \times \left(\frac{\sqrt{2}-1}{2\sqrt{3}}\right)^4 = \frac{(\sqrt{2}-1)^4}{144}$$

Illustration 47

The first term of a G.P. is 1. The sum of the third and fifth terms is 90. Find the common ratio of the G.P.

Let r be the common ratio of the G.P. It is given that the first term a = 1.

Now, $a_3 + a_5 = 90$

$\Rightarrow$ $ar^2 + ar^4 = 90$

$\Rightarrow$ $r^2 + r^4 = 90 \Rightarrow r^4 + r^2 - 90 = 0 \Rightarrow r^4 + 10r^2 - 9r^2 - 90 = 0$

$\Rightarrow$ $(r^2 + 10)(r^2 - 9) = 0$ $\Rightarrow$ $r^2 - 9 = 0 \Rightarrow r = \pm 3$.

Illustration 48

If the first and the nth terms of a G.P. are a and b respectively and if P is the product of the first n terms, prove that $P^2 = (ab)^n$.

Let r be the common ratio of the given G.P. Then,

$$b = \text{nth term} = ar^{n-1} \Rightarrow r^{n-1} = \frac{b}{a} \Rightarrow r = \left(\frac{b}{a}\right)^{\frac{1}{n-1}}$$

Now, P = Product of the first n terms

$\Rightarrow \quad P = a \,.\, ar.\, ar^2 \ldots ar^{n-1}$

$\Rightarrow \quad P = a^n\, r^{1+2+3+\ldots+(n-1)}$

$$\Rightarrow \quad P = a^n\, r^{\frac{n(n-1)}{2}} \qquad \left[\because 1+2+3+\ldots+(n-1) = \left(\frac{n-1}{2}\right)(1+(n-1)) = \frac{n(n-1)}{2}\right]$$

$$\Rightarrow \quad P = a^n\left\{\left(\frac{b}{a}\right)^{\frac{1}{n-1}}\right\}^{\frac{n(n-1)}{2}} = a^n\left(\frac{b}{a}\right)^{n/2} = a^{n/2}b^{n/2} = (ab)^{n/2}$$

$\therefore \quad P^2 = [(ab)^{n/2}]^2 = (ab)^n$

Illustration 49

The third term of a G.P. is 4. Find the product of its first five terms.

Let a be the first term and r be the common ratio. Then,

$a_3 = 4 \Rightarrow ar^2 = 4$ ....(i)

Product of first five terms $= a_1\, a_2\, a_3\, a_4\, a_5$

$= a(ar)\,(ar^2)\,(ar^3)\,(ar^4)$

$= a^5\, r^{10} = (ar^2)^5 = 4^5$ [Using (i)]

Illustration 50

If the pth, qth and rth terms of a G.P. are a, b, c respectively. Prove that :

$a^{(q-r)} \,.\, b^{(r-p)} \,.\, c^{(p-q)} = 1$

Let A be the first term and R be the common ratio of the given G.P. Then,

a = pth term $\Rightarrow a = AR^{(p-1)}$, b = qth term $\Rightarrow b = AR^{(q-1)}$,

and, c = rth term $\Rightarrow c = AR^{(r-1)}$

Now, $a^{(q-r)} \,.\, b^{(r-p)} \,.\, c^{(p-q)}$

$= \left\{AR^{(p-1)}\right\}^{(q-r)} . \left\{AR^{(q-1)}\right\}^{(r-p)} . \left\{AR^{(r-1)}\right\}^{(p-q)}$

$= A^{(q-r)}\, R^{(p-1)(q-r)} \,.\, A^{(r-p)}\, R^{(q-1)(r-p)} \,.\, A^{(p-q)}\, R^{(r-1)(p-q)}$

$= A^{(q-r+r-p+p-q)}\, R^{(p-1)(q-r)+(q-1)(r-p)+(r-1)(p-q)}$

$= A^0\, R^{p(q-r)+q(r-p)+r(p-q)-(q-r)-(r-p)-(p-q)} = A^0R^0 = 1.$

Illustration 51

If a, b, c are respectively the pth, qth and rth terms of a G.P., show that

$(q - r) \log a + (r - p) \log b + (p - q) \log c = 0$.

Let A be the first term and R the common ratio of the given G.P. Then,

a = pth term $\Rightarrow a = AR^{p-1} \Rightarrow \log a = \log A + (p - 1) \log R$ ……(i)

b = qth term $\Rightarrow b = AR^{q-1} \Rightarrow \log b = \log A + (q - 1) \log R$ ……(ii)

c = rth term $\Rightarrow c = AR^{r-1} \Rightarrow \log c = \log A + (r - 1) \log R$ ……(iii)

Now, $(q - r) \log a + (r - p) \log b + (p - q) \log c$

$= (q - r) \{\log A + (p - 1) \log R\} + (r - p) \{\log A + (q - 1) \log R\} + (p - q) \{\log A + (r - 1) \log R\}$

[Using (i), (ii) and (iii)]

$= \log A \{(q - r) + (r - p)+(p - q)\}+ \log R \{(p -1)(q - r)+(q - 1) (r - p) + (r - 1) (p - q)\}$

$= \log A. 0 + \log R \{p (q - r) + q(r - p) + r(p - q) - (q - r) - (r - p) - (p - q)= \log A. 0 + \log R . 0 = 0$.

Illustration 52

Find all sequences which are simultaneously an A.P. and a G.P.

Let $< a_n >$ be a sequence which is both an A.P. and a G.P. Let a_n, a_{n+1}, a_{n+2} be three consecutive terms of the A.P., then

$2 a_{n+1} = a_n + a_{n+2}, n \geq 1$ ….(i)

Let r be the common ratio of the sequence when it is considered a G.P. Then,

$a_n = a_1 r^{n-1}$, $a_{n+1} = a_1 r^n$ and $a_{n+2} = a_1 r^{n+1}$

Putting these values in (i), we get $2 a_1 r^n = a_1 r^{n-1} + a_1 r^{n+1}$

$\Rightarrow$ $2r = 1 + r^2 \Rightarrow r^2 - 2r + 1 = 0 \Rightarrow (r - 1)^2 = 0 \Rightarrow r = 1$

So , $a_1 = a_2 = a_3 = ….$ i.e. the constant sequence is the only sequence which is both an A.P. and a G.P.

Illustration 53

In a finite G.P. the product of the terms equidistant from the beginning and the end is always same and equal to the product of first and last term.

Let $a_1, a_2, a_3, …, a_{n-1}, a_n$ be a finite G.P. with common ratio r. Then, for any k,

a_k = kth term from the beginning = $a_1 r^{k-1}$.

a_{n-k+1} = kth term from the end = $a_n \left(\frac{1}{r}\right)^{k-1}$, where $1 < k < n$.

Now, $a_k . a_{n-k+1} = a_1 r^{k-1} a_n \left(\frac{1}{r}\right)^{k-1} = a_1 a_n$ for all $1 < k < n$.

Selection of terms in G.P.

Sometimes it is required to select a finite number of terms in G.P. It is always convenient if we select the terms in the following manner.

No. of terms **Terms** **Common ratio**

3	$\frac{a}{r}, a, ar$	r
4	$\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$	r^2
5	$\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$	r

If the product of the numbers is not given, then the numbers are taken as a, ar, ar^2, ar^3, ….

Illustration 54

If the sum of three numbers in G.P. is 38 and their product is 1728 , find them.

Let the numbers be $\frac{a}{r}$, a, ar. Then,

Product = 1728 $\Rightarrow \frac{a}{r} . a. ar = 1728 \Rightarrow a^3 = 1728 \Rightarrow a = 12$

Sum = 38

$\Rightarrow \quad \frac{a}{r} + a + ar = 38$

$\Rightarrow \quad a\left(\frac{1}{r}+1+r\right) = 38$

$\Rightarrow \quad 12\left(\frac{1+r+r^2}{r}\right) = 38 \Rightarrow 6 + 6r + 6r^2 = 19r.$

$\Rightarrow \quad 6r^2 - 13r + 6 = 0 \Rightarrow (3r - 2)(2r - 3) = 0 \Rightarrow r = 3/2$ or $r = 2/3$

Hence, putting the values of a and r, the required numbers are 8, 12, 18 or 18, 12, 8.

Illustration 55

If the continued product of three numbers in G.P. is 216 and the sum of their products in pairs is 156, find the numbers.

Let the three numbers be a/r, a, ar. Then,

Product = 216 $\Rightarrow$ a/r . a. ar = 216 $\Rightarrow a^3 = 216 \Rightarrow a = 6$

Sum of the products in pairs = 156

$\Rightarrow \quad \frac{a}{r}.a + a.ar + \frac{a}{r}.ar = 156$

$\Rightarrow \quad a^2\left(\frac{1}{r}+r+1\right) = 156$

$\Rightarrow \quad 36\left(\frac{1+r^2+r}{r}\right) = 156$

$\Rightarrow \quad 3(r^2 + r + 1) = 13r \Rightarrow 3r^2 - 10r + 3 = 0$

$\Rightarrow \quad (3r - 1)(r - 3) = 0 \Rightarrow r = \frac{1}{3}$ or $r = 3$

Hence , putting the values of a and r, the required numbers are 18, 6, 2 or 2, 6, 18.

Illustration 56

Find three numbers in G.P. whose sum is 13 and the sum of whose squares is 91.

Let the three numbers be a, ar, ar^2 . Then,

Sum = 13 $\Rightarrow$ a + ar + ar^2 = 13 $\Rightarrow$ $a(1 + r + r^2) = 13$ ....(i)

Sum of the squares = 91

$\Rightarrow$ $a^2 + a^2 r^2 + a^2r^4 = 91 \Rightarrow a^2(1 + r^2 + r^4) = 91$ ....(ii)

Now, $a(1 + r + r^2) = 13$

$\Rightarrow$ $a^2 (1 + r + r^2)^2 = 169$

$\Rightarrow$ $a^2(1 + r^2 + r^4) + 2a^2 r (1 + r + r^2) = 169$

$\Rightarrow$ $91 + 2ar \{a (1 + r + r^2)\} = 169$

$\Rightarrow$ $91 + 2ar \times 13 = 169$ [Using (ii)]

$\Rightarrow$ ar = 3 [Using (i)]

$\Rightarrow$ $a = \frac{3}{r}$ ...(iii)

Putting $a = \frac{3}{r}$ in (i), we get

$$\frac{3}{r}(1 + r + r^2) = 13$$

$\Rightarrow$ $\frac{3}{r} + 3 + 3r = 13 \Rightarrow 3r^2 - 10r + 3 = 0$

$\Rightarrow$ $(3r - 1)(r - 3) = 0 \Rightarrow r = 3$ or, $r = \frac{1}{3}$

Putting r = 3 in (iii) , we get a = 1 . So, the numbers are 1, 3, 9.

Putting $r = \frac{1}{3}$ in (iii) , we get a = 9. So, the numbers are 9, 3, 1.

Hence , the numbers are 1, 3, 9 or 9, 3, 1.

Illustration 57

Find four numbers in G.P. whose sum is 85 and product is 4096.

Let the four numbers in G.P. be $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$

Product = 4096 $\Rightarrow a^4 = 4096 \Rightarrow a^4 = 8^4 \Rightarrow a = 8$

Sum = 85 $\Rightarrow a\left(\frac{1}{r^3} + \frac{1}{r} + r + r^3\right) = 85 \Rightarrow 8\left(r^3 + \frac{1}{r^3}\right) + 8\left(r + \frac{1}{r}\right) = 85$

$\Rightarrow$ $8\left[\left(r + \frac{1}{r}\right)^3 - 3\left(r + \frac{1}{r}\right)\right] + 8\left(r + \frac{1}{r}\right) = 85 \Rightarrow$ $8\left(r + \frac{1}{r}\right)^3 - 16\left(r + \frac{1}{r}\right) - 85 = 0$

$\Rightarrow$ $8x^3 - 16x - 85 = 0$, where $r + \frac{1}{r} = x$

$\Rightarrow$ $(2x - 5)(4x^2 + 10x + 17) = 0$

$\Rightarrow$ $2x - 5 = 0$ [$\because$ $4x^2 + 10x + 17 = 0$ has imaginary roots]

$\Rightarrow \quad x = (5/2)$

$\Rightarrow \quad r + \frac{1}{r} = \frac{5}{2} \Rightarrow 2r^2 - 5r + 2 = 0 \Rightarrow (r-2)(2r-1) = 0 \Rightarrow r = 2 \text{ or } r = \frac{1}{2}$

Putting a = 8 and r = 2 or r = $\frac{1}{2}$, we obtain that the four numbers are either 1, 4, 16, 64 or 64, 16, 4, 1.

Illustration 58

Three numbers are in G.P. whose sum is 70. If the extremes be each multiplied by 4 and the means by 5, they will be in A.P. Find the numbers.

Let the numbers be a, ar, ar^2.

Sum = 70 $\Rightarrow a(1 + r + r^2) = 70$ ....(i)

It is given that 4a, 5ar, $4ar^2$ are in A.P.

$\therefore \quad 2(5ar) = 4a + 4ar^2$

$\Rightarrow \quad 5r = 2 + 2r^2$

$\Rightarrow \quad 2r^2 - 5r + 2 = 0 \Rightarrow (2r-1)(r-2) = 0 \Rightarrow r = 2 \text{ or, } r = 1/2.$

Putting r = 2 in (i), we obtain a = 10. Putting r = 1/2 in (i), we get a = 40.

Hence , the numbers are 10, 20, 40 or 40, 20, 10.

Illustration 59

Find three numbers in G.P. whose sum is 52 and the sum of whose products in pairs is 624.

Let the required numbers be a, ar, ar^2. Then,

Sum = 52 $\Rightarrow a + ar + ar^2 = 52 \Rightarrow a(1 + r + r^2) = 52.$ ....(i)

Sum of the products in pairs = 624

$\Rightarrow \quad a \cdot ar + ar \cdot ar^2 + a \cdot ar^2 = 624$ ....(ii)

$\Rightarrow \quad a^2r(1 + r + r^2) = 624$

Dividing (ii) by (i), we get

$ar = 12 \Rightarrow a = \frac{12}{r}$ ....(iii)

Putting a = $\frac{12}{r}$ in (i), we get

$\frac{12}{r}(1 + r + r^2) = 52$

$\Rightarrow \quad 3r^2 - 10r + 3 = 0$

$\Rightarrow \quad (3r-1)(r-3) = 0 \Rightarrow r = 1/3 \text{ or, } r = 3$

From (iii), r = 3 $\Rightarrow$ a = 4 and r = 1/3 $\Rightarrow$ a = 36

Hence , the numbers are 4, 12, 36 or 36, 12, 4

Illustration 60

The product of first three terms of a G.P. is 1000. If 6 is added to its second term and 7 added to its third term, the terms become in A.P. Find the G.P.

Let the terms of the given G.P. be $\frac{a}{r}$, a, ar

Then, product = 1000 $\Rightarrow a^3 = 1000 \Rightarrow a = 10$.

It is given that $\frac{a}{r}$, a + 6, ar + 7 are in A.P.

$$2(a + 6) = \frac{a}{r} + ar + 7$$

$$\Rightarrow \quad 32 = \frac{10}{r} + 10r + 7 \Rightarrow 25 = \frac{10}{r} + 10r$$

$$\Rightarrow \quad 5 = \frac{2}{r} + 2r \Rightarrow 2r^2 - 5r + 2 = 0 \Rightarrow (2r - 1)(r - 2) = 0 \Rightarrow r = 2, 1/2 .$$

Hence, the G.P. is 5, 10, 20 , …. or 20, 10, 5, ….

Illustration 61

Find four numbers in G.P. in which the third term is greater than the first term by 9 and the second term is greater than the fourth term by 18.

Let the four numbers be a, ar, ar^2, ar^3. It is given that

$$ar^2 - a = 9 \quad \text{and} \quad ar - ar^3 = 18$$

$$\Rightarrow \quad a(r^2 - 1) = 9 \quad \text{and} \quad ar(1 - r^2) = 18$$

$$\Rightarrow \quad \frac{ar(1-r^2)}{a(r^2-1)} = \frac{18}{9} \Rightarrow r = -2$$

Putting r = –2 in a $(r^2 - 1) = 9$, we get a = 3. Hence , the numbers are 3, –6, 12, –24.

Properties of Geometric Progression

Prop I **If all the terms of a G.P. be multiplied or divided by the same non-zero constant, then it remains a G.P. with the same common ratio.**

Proof : Let $a_1, a_2, a_3, \ldots, a_n, \ldots$ be a G.P. with common ratio r.

Then, $\frac{a_{n+1}}{a_n} = r$, for all $n \in N$ ....(i)

Let k be a non-zero constant. Multiplying all the terms of the given G.P. by k, we obtain the new sequence.

$$ka_1, ka_2, ka_3, \ldots ka_n, \ldots$$

Clearly, $\frac{ka_{n+1}}{ka_n} = \frac{a_{n+1}}{a_n} = r$ for all $n \in N$.

Hence, the new sequence also forms a G.P. with common ratio r.

Prop II. **The reciprocals of the terms of a given G.P. form a G.P.**

Proof : Let $a_1, a_2, a_3, \ldots, a_n, \ldots$ be a G.P. with common ratio r. Then,

$\frac{a_{n+1}}{a_n} = r$ for all $n \in N$. ....(i)

The sequence formed by the reciprocals of the terms of the given G.P. is

$$\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \ldots \frac{1}{a_n}, \ldots$$

We have, $\dfrac{1/a_{n+1}}{1/a_n} = \dfrac{a_n}{a_{n+1}} = \dfrac{1}{r}$ [Using (i)]

So, the new sequence is a G.P. with common ratio 1/ r.

Prop III. **If each term of a G.P. be raised to the same power, the resulting sequence also forms a G.P.**

Proof : Let $a_1, a_2, a_3, \ldots a_n, \ldots$ be a G.P. with common ratio r. Then,

$\dfrac{a_{n+1}}{a_n} = r$ for all $n \in N$ ....(i)

Let k be a non-zero real number. Consider the sequence

$$a_1^k, a_2^k, a_3^k, \ldots\ldots, a_n^k, \ldots$$

We have, $\dfrac{a_{n+1}^k}{a_n^k} = \left(\dfrac{a_{n+1}}{a_n}\right)^k = r^k$ for all $n \in N$. [Using (i)]

Hence, $a_1^k, a_2^k, a_3^k, \ldots\ldots, a_n^k, \ldots$ is a G.P. with common ratio r^k.

Prop IV. **In a finite G.P. the product of the terms equidistant from the beginning and the end is always same and is equal to the product of the first and the last term.**

Proof : Let $a_1, a_2, a_3, \ldots, a_n$ be a finite G.P. with common ratio r. Then,

kth term from the beginning = $a_k = a_1 r^{k-1}$

kth term from the end = (n – k + 1)th term from the beginning

$= a_{n-k+1} = a_1 r^{n-k}$

$\therefore$ (kth term from the beginning) (kth term from the end)

$= a_k a_{n-k+1}$

$= a_1 r^{k-1} a_1 r^{n-k} = a_1^2 r^{n-1} = a_1 . a_1 r^{n-1} = a_1 a_n$ for all k = 2, 3, …, n – 1

Hence, the product of the terms equidistant from the beginning and the end is always same and is equal to the product of the first and the last term.

Prop V. **Three non-zero numbers a, b, c are in G.P. iff $b^2 = ac$**

Proof : a, b, c are in G.P.

$\Leftrightarrow \dfrac{b}{a} = \dfrac{c}{b}$ = (common ratio) $\Leftrightarrow b^2 = ac$

NOTE : When a, b, c are in G.P. then b is known as the Geometric mean of a and c.

Prop VI. **If the terms of a given G.P. are chosen at regular intervals, then the new sequence so formed also forms a G.P.**

Prop VII. **If $a_1, a_2, a_3, \ldots, a_n, \ldots$ is a G.P. of non-zero non-negative terms, then $\log a_1, \log a_2, \ldots, \log a_n, \ldots$ is an A.P. and vice – versa.**

Proof : Let $a_1, a_2, a_3, \ldots, a_n, \ldots$ be a G.P. of non-zero non-negative terms with common ratio r, Then,

$a_n = a_1 r^{n-1}$, for all $n \in N$

$\Rightarrow$ $\log a_n = \log a_1 + (n-1) \log r$, for all $n \in N$.

Let $b_n = \log a_n = \log a_1 + (n-1) \log r$, for all $n \in N$

Then, $b_{n+1} - b_n = [\log a_1 + n \log r] - [\log a_1 + (n-1) \log r] = \log r$ for all $n \in N$

Clearly, $b_{n+1} - b_n = \log r$ = constant for all $n \in N$.

Hence, $b_1, b_2, \ldots, b_n, \ldots$ i.e. $\log a_1, \log a_2, \ldots, \log a_n, \ldots$ is an A.P. with common difference $\log r$.

Conversely, let $\log a_1, \log a_2, \ldots, \log a_n, \ldots$ be an A.P. with common difference d. Then, $\log a_{n+1} - \log a_n = d$, for all $n \in N$.

$\Rightarrow$ $\log\left(\frac{a_{n+1}}{a_n}\right) = d$, for all $n \in N$

$\Rightarrow$ $\frac{a_{n+1}}{a_n} = e^d$, for all $n \in N$.

$\Rightarrow$ $a_1, a_2, a_3, \ldots, a_n, \ldots$ is a G.P. with common ratio e^d.

Illustration 62

If a, b, c, d are in G.P. show that :

(i) $(b-c)^2 + (c-a)^2 + (d-b)^2 = (a-d)^2$

(ii) $(ab + bc + cd)^2 = (a^2 + b^2 + c^2)(b^2 + c^2 + d^2)$

Let r be the common ratio of the G.P. a, b, c, d . Then, $b = ar$, $c = ar^2$ and $d = ar^3$.

(i) $\therefore$ LHS $= (b-c)^2 + (c-a)^2 + (d-b)^2 = (ar - ar^2)^2 + (ar^2 - a)^2 + (ar^3 - ar)^2$

$= a^2r^2 (1-r)^2 + a^2 (r^2-1)^2 + a^2r^2 (r^2-1)^2$

$= a^2(r^6 - 2r^3 + 1) = a^2 (1-r^3)^2 = (a-d)^2$ = RHS

(ii) LHS $= (ab + bc + cd)^2 = (a \cdot ar + ar \cdot ar^2 + ar^2 \cdot ar^3)^2 = a^4r^2 (1 + r^2 + r^4)^2$

RHS $= (a^2 + b^2 + c^2)(b^2 + c^2 + d^2) = (a^2 + a^2r^2 + a^2r^4)(a^2r^2 + a^2r^4 + a^2r^6)$

$= a^2 (1 + r^2 + r^4) \cdot a^2r^2 (1 + r^2 + r^4) = a^4r^2 (1 + r^2 + r^4)^2$

$\therefore$ LHS = RHS.

Illustration 63

If a, b, c, d are in G.P. , prove that a + b, b + c, c + d are also in G.P.

Let r be the common ratio of the G.P. a, b, c, d. Then $b = ar$, $c = ar^2$ and $d = ar^3$.

$\therefore$ $a + b = a + ar = a(1 + r)$; $b + c = ar + ar^2 = ar(1 + r)$

and $c + d = ar^2 + ar^3 = ar^2 (1 + r)$

Now, $(b + c)^2 = [ar(1 + r)]^2 = a^2r^2 (1 + r)^2 = [a(1 + r)][ar^2 (1 + r)]$

$= (a + b)(c + d)$ $[\because a + b = a(1 + r)$ and $c + d = ar^2(1 + r)]$

Hence, a + b, b + c, c + d are in G.P.

Illustration 64

If $a^2 + b^2$, $ab + bc$ and $b^2 + c^2$ are in G.P., prove that a, b, c are also in G.P.

$a^2 + b^2$, $ab + bc$, $b^2 + c^2$ are in G.P.

$\Rightarrow \quad (ab + bc)^2 = (a^2 + b^2)(b^2 + c^2)$

$\Rightarrow \quad a^2b^2 + b^2c^2 + 2ab^2c = a^2b^2 + a^2c^2 + b^2c^2 + b^4$

$\Rightarrow \quad b^4 + a^2c^2 - 2ab^2c = 0$

$\Rightarrow \quad (b^2 - ac)^2 = 0$

$\Rightarrow \quad b^2 = ac \Rightarrow$ a, b, c are in G.P.

Illustration 65

If a, b, c are in G.P. , prove that $\frac{1}{a+b}, \frac{1}{2b}, \frac{1}{b+c}$ are in A.P.

Since a, b, c are in G.P. we have

$\frac{b}{a} = \frac{c}{b} = r$, where r is the common ratio

$\therefore$ $b = ar$ and $c = br = ar.\, r = ar^2$

x, y, z will be in A.P. if $y - x = z - y$, i.e. if $x + z = 2y$

We have

$$\frac{1}{a+b} = \frac{1}{a+ar} = \frac{1}{a(1+r)}, \frac{1}{2b} = \frac{1}{2ar}$$

and $$\frac{1}{b+c} = \frac{1}{ar+ar^2} = \frac{1}{ar(1+r)}$$

$$\therefore \quad \frac{1}{a+b} + \frac{1}{b+c} = \frac{1}{a(1+r)} + \frac{1}{ar(1+r)} = \frac{r+1}{ar(1+r)} = \frac{1}{ar} = \frac{1}{b} = \frac{2}{2b}$$

This shows that $\frac{1}{a+b}, \frac{1}{2b}, \frac{1}{b+c}$ are in A.P.

To find the sum of first n terms of a G.P.

Let a be the first term and r the common ratio of a G.P. and let S_n denote the sum of the first n terms of the G.P. Then

$S_n = a + ar + ar^2 + \ldots.. + ar^{n-1}$

and $r.\, S_n = ar + ar^2 + \ldots. + ar^{n-1} + ar^n$

By subtracting, $S_n - r.S_n = a - ar^n$, or, $S_n(1 - r) = a(1 - r^n)$,

$$\therefore \quad S_n = \frac{a(1-r^n)}{1-r}, \text{ or, } \frac{a(r^n-1)}{r-1}$$

If $r > 1$, we write $S_n = \frac{a(r^n-1)}{r-1}$

and if $r < 1$, we write $S_n = \frac{a(1-r^n)}{1-r}$

Note : The above formula for S_n fails when r = 1 , as the form $\frac{0}{0}$ is undefined. However,

S_n = a + a + a + …. to n terms = na in this case.

Illustration 66

If $\{a_n\}$ is a G.S. (i.e., Geometric sequence) and $a_1 = 4$, r = 5, find a_6 and S_6.

Here a_1 = 1st term = 4, r = C.R. = 5, a_n = nth term = $a_1(r)^{n-1}$.

∴ a_6 = 6th term = $a_1\,(r)^{6-1} = 4.(5)^5 = 4 \times 3125 = 12500$.

S_6 = Sum of 1st 6 terms = $\frac{a_1(r^6-1)}{r-1} = \frac{4(5^6-1)}{5-1}$

$= \frac{4.(15625-1)}{4} = 15624$.

Illustration 67

Find the sum of each of the following series :

(i) 1 + 3 + 9 + 27 + ….. to 10 terms ;

(ii) $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}$ + ….. to 12 terms.

(i) The series is a G.P. with common ratio 3.

For sum in G.P. : $S_n = \frac{a(r^n-1)}{r-1}$, where r > 1

Here a = 1, r = 3, n = 10, S_{10} = ?

We have

$$S_{10} = \frac{1.(3^{10}-1)}{3-1} = \frac{59049-1}{2} = 29524$$

(ii) Here a = 1, r = $\frac{1}{2}$ (< 1) and n = 12.

For sum in G.P., we have $S_n = \frac{a(1-r^n)}{1-r}$, where r < 1.

$$\therefore\ S_{12} = \frac{1.\left\{1-\left(\frac{1}{2}\right)^{12}\right\}}{1-\frac{1}{2}} = \frac{1-\frac{1}{2^{12}}}{\frac{1}{2}} = \frac{1-\frac{1}{4096}}{\frac{1}{2}} = \frac{4095}{4096}\times\frac{2}{1} = \frac{4095}{2048} = 1\frac{2047}{2048}$$

Illustration 68

If you save 1 paise today, 2 paise the next day, 4 paise the succeeding day and so on, what will be your total savings in two weeks ?

Total savings in two weeks.

= (1 + 2 + 4 + 8 + …. to 14 terms) paise.

$= \frac{1.(2^{14}-1)}{2-1}$ [∵ a = 1, r = 2, n = 14]

$= \dfrac{16384 - 1}{1}$ = 16383 paise = Rs. 163.83.

Illustration 69

How many terms of the G.P. 3, $\dfrac{3}{2}, \dfrac{3}{4}$, ……. are needed to give sum $\dfrac{3069}{512}$?

For the given G.P., a = 1st term = 3, r = common ratio = $\dfrac{1}{2}$

Let n be the required number of terms. Then $S_n = \dfrac{3069}{512}$

or, $\dfrac{a(1-r^n)}{1-r} = \dfrac{3069}{512}$, or, $\dfrac{3\left\{1-\left(\frac{1}{2}\right)^n\right\}}{1-\frac{1}{2}} = \dfrac{3069}{512}$

or, $3 \times 2\left\{1-\left(\dfrac{1}{2}\right)^n\right\} = \dfrac{3069}{512}$, or, $1 - \dfrac{1}{2^n} = \dfrac{1023}{1024}$

or, $\dfrac{1}{2^n} = 1 - \dfrac{1023}{1024} = \dfrac{1}{1024}$, or, $2^n = 1024 = 2^{10}$;

$\therefore$ n = 10 . Hence 10 terms of the G.P. are needed.

Illustration 70

(i) Sum to n terms : 4 + 44 + 444 + …………

(ii) 7 + 77 + 777 + ……

(i) S_n = 4 + 44 + 444 + ……. to n terms

= 4(1 + 11 + 111 + …… to n terms)

$= \dfrac{4}{9}$(9 + 99 + 999 + ….. to n terms)

$= \dfrac{4}{9}$\{(10 – 1) + (100 – 1) + (1000 – 1) + …. to n groups)\}

$= \dfrac{4}{9}$\{(10 + 100 + 1000 + ….. to n terms) – (1 + 1 + 1 + ….. to n terms)\}

$= \dfrac{4}{9}\left\{\dfrac{10.(10^n - 1)}{10-1} - n\right\}$, using $S_n = \dfrac{a(r^n-1)}{r-1}$ $= \dfrac{4}{9}\left\{\dfrac{10}{9}(10^n - 1) - n\right\}$

(ii) Exactly similar to (i) , $S_n = \dfrac{7}{9}\left\{\dfrac{10}{9}(10^n - 1) - n\right\}$

Illustration 71

(i) Find the sum to 'n' terms of .8 + .88 + .888 + ……

(ii) 0.6 + 0.66 + 0.666 + ……….

(i) We have

S_n = .8 + .88 + .888 + ……. to n terms

$= 8(.1 + .11 + .111 + \ldots\ \text{to n terms})$

$= \frac{8}{9}(.9 + .99 + .999 + \ldots\ \text{to n terms})$

$= \frac{8}{9}[(1 - .1) + (1 - .01) + (1 - .001) + \ldots\ \text{to n groups}]$

$= \frac{8}{9}[(1 + 1 + 1 + \ldots\ \text{to n terms}) - (.1 + .01 + .001 + \ldots\ \text{to n terms})]$

$= \frac{8}{9}\left[n - \frac{.1\{1-(.1)^n\}}{1-.1}\right]$ using $S_n = \frac{a(1-r^n)}{1-r}$

$= \frac{8}{9}\left[n - \frac{.1}{.9}\{1-(.1)^n\}\right] = \frac{8}{9}\left[n - \frac{1}{9}\{1-(.1)^n\}\right]$

(ii) Exactly similar to (i) ; $S_n = \frac{2}{3}\left[n - \frac{1}{9}\{1-(.1)^n\}\right]$

Illustration 72

If S be the sum, P the product and R the sum of the reciprocals of n terms in G.P. prove that $\left(\frac{S}{R}\right)^n = P^2$.

Let a be the first term and r be the common ratio of the G.P.

Then $S = a + ar + ar^2 + \ldots + ar^{n-1} = \frac{a.(r^n - 1)}{r - 1}$

$P = a \times ar \times ar^2 \times \ldots \times ar^{n-1}$

$= a^n.\ r^{1+2+3+\ldots+(n-1)}$

$= a^n.\ r^{\frac{(n-1)(n-1+1)}{2}} = a^n.\ r^{\frac{n(n-1)}{2}}$

and $R = \frac{1}{a} + \frac{1}{ar} + \frac{1}{ar^2} + \ldots + \frac{1}{ar^{n-1}}$

$= \frac{\frac{1}{a}\left\{1-\left(\frac{1}{r}\right)^n\right\}}{1-\frac{1}{r}} = \frac{\frac{1}{a}\times\left(1-\frac{1}{r^n}\right)}{\frac{r-1}{r}} = \frac{1}{a}\times\frac{r^n-1}{r^n}\times\frac{r}{r-1}$

$= \frac{r(r^n-1)}{ar^n(r-1)} = \frac{r^n-1}{ar^{n-1}(r-1)}$

$\therefore \quad \frac{S}{R} = \frac{a(r^n-1)}{r-1}\times\frac{ar^{n-1}(r-1)}{r^n-1} = a^2.\ r^{n-1}$

$\therefore \quad \left(\frac{S}{R}\right)^n = (a^2.\ r^{n-1})^n = a^{2n}.\ r^{n(n-1)}$

and $P^2 = \left[a^n . r^{\frac{n(n-1)}{2}}\right]^2 = a^{2n} . r^{n(n-1)}$

Hence $\left(\frac{S}{R}\right)^n = P^2$.

Illustration 73

Prove that the sum to n terms of the series 11 + 103 + 1005 + ... is $\frac{10}{9}(10^n - 1) + n^2$.

Let S_n denote the sum to n terms of the given series. Then,

$S_n = 11 + 103 + 1005 + ...$ to n terms.

$\Rightarrow \quad S_n = (10 + 1) + (10^2 + 3) + (10^3 + 5) + ... + \{10^n + (2n - 1)\}$

$\Rightarrow \quad S_n = (10 + 10^2 + ... + 10^n) + ((1 + 3 + 5 + ... + (2n - 1))$

$\Rightarrow \quad S_n = \frac{10(10^n - 1)}{(10 - 1)} + \frac{n}{2}(1 + 2n - 1) = \frac{10}{9}(10^n - 1) + n^2$

Illustration 74

Determine the number of terms in G.P. < a_n >, if $a_1 = 3$, $a_n = 96$ and $S_n = 189$.

Let r be the common ratio of the given G.P. Then,

$a_n = 96 \Rightarrow a_1 r^{n-1} = 96 \Rightarrow 3r^{n-1} = 96 \Rightarrow r^{n-1} = 32$ ....(i)

Now, $S_n = 189$

$\Rightarrow \quad a_1\left(\frac{r^n - 1}{r - 1}\right) = 189$

$\Rightarrow \quad 3\left(\frac{(r^{n-1})r - 1}{r - 1}\right) = 189$

$\Rightarrow \quad 3\left(\frac{32r - 1}{r - 1}\right) = 189$

$\Rightarrow \quad 32r - 1 = 63r - 63 \Rightarrow 31r = 62 \Rightarrow r = 2$

Putting r = 2 in (i), we get

$2^{n-1} = 32 \Rightarrow 2^{n-1} = 2^5 \Rightarrow n - 1 = 5 \Rightarrow n = 6$.

Illustration 75

In an increasing G.P. , the sum of the first and the last term is 66, the product of the second and the last but one is 128 and the sum of the terms is 126. How many terms are there in the progression ?

Let a be the first term and r the common ratio of the given G.P. Further, let there be n terms in the given G.P. Then,

$a_1 + a_n = 66 \Rightarrow a + ar^{n-1} = 66$ ...(i)

$a_2 . a_{n-1} = 128$

$\Rightarrow \quad ar . ar^{n-2} = 128$

$\Rightarrow \quad a^2 r^{n-1} = 128 \Rightarrow a \,.\, (ar^{n-1}) = 128 \Rightarrow ar^{n-1} = \dfrac{128}{a}$

Putting this value of ar^{n-1} in (i), we get

$$a + \frac{128}{a} = 66 \Rightarrow a^2 - 66a + 128 = 0 \Rightarrow (a-2)(a-64) = 0 \Rightarrow a = 2, 64$$

Putting a = 2 in (i), we get

$$2 + 2 \,.\, r^{n-1} = 66 \Rightarrow r^{n-1} = 32$$

Putting a = 64 in (i), we get

$$64 + 64r^{n-1} = 66 \Rightarrow r^{n-1} = \frac{1}{32}$$

We reject the second value as the G.P. is an increasing G.P. and therefore, r > 1.

Now, $\quad S_n = 126$

$\Rightarrow \quad 2\left(\dfrac{r^n - 1}{r-1}\right) = 126 \qquad \Rightarrow \qquad \dfrac{r^n - 1}{r-1} = 63$

$\Rightarrow \quad \dfrac{r^{n-1}.r - 1}{r-1} = 63 \Rightarrow \qquad \dfrac{32r - 1}{r-1} = 63 \Rightarrow r = 2$

$\therefore \quad r^{n-1} = 32 \Rightarrow 2^{n-1} = 32 = 2^5 \Rightarrow n - 1 = 5 \Rightarrow n = 6$

PRACTICE ASSIGNMENT III

1. Find the 20th and *n*th terms of the G.P. $\dfrac{5}{2}, \dfrac{5}{4}, \dfrac{5}{8}, \ldots\ldots$
2. Find the 12th term of a G.P. whose 8th term is 192 and the common ratio is 2.
3. The 5th, 8th and 11th terms of a G.P. are *p, q* and *s*, respectively. Show that $q^2 = ps$.
4. The 4th term of a G.P. is square of its second term, and the first term is – 3. Determine its 7th term.
5. The second term of a G.P. is $\dfrac{5}{4}$ and its fourth term is $\dfrac{1}{5}$. Find its third term.
6. Which term of the following sequences :

   (i) $2, 2\sqrt{2}, 4, \ldots$ is 128 ? (ii) $\sqrt{3}, 3, 3\sqrt{3}, \ldots$ is 729 ? (iii) $\dfrac{1}{3}, \dfrac{1}{9}, \dfrac{1}{27}, \ldots$ is $\dfrac{1}{19683}$?
7. For what values of x, the numbers $-\dfrac{2}{7}, x, -\dfrac{2}{7}$ are in G.P. ?
8. Find the value of n, if the nth terms of the series 5 + 10 + 20 +………and 1280 + 640 + 320 + ….are equal.
9. If a, b, c are in G.P; show that

   $$a^2b^2c^2\left(\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3}\right) = a^3 + b^3 + c^3$$
10. If a, b, c, d are in G.P, prove that

    (i) $(a^2 + b^2), (b^2 + c^2), (c^2 + d^2)$ are in G.P.

(ii) $(a^n + b^n), (b^n + c^n), (c^n + d^n)$ are in G.P.

(iii) $(a^2 - b^2)$, $(b^2 - c^2)$, $(c^2 - d^2)$ are in G.P.

(iv) $\frac{1}{a^2+b^2}, \frac{1}{b^2+c^2}, \frac{1}{c^2+d^2}$ are in G.P.

11. Find the sum to indicated number of terms in each of the following G.P.

(i) 0.15, 0.015, 0.0015, … 20 terms.

(ii) $\sqrt{7}$, $\sqrt{21}$, $3\sqrt{7}$, …… *n* terms.

(iii) 1, $-a$, $a^2, -a^3$,…. *n* terms (if $a \neq -1$).

(iv) $x^3, x^5, x^7, \ldots . n$ terms (if $x \neq \pm 1$).

12. The sum of first three terms of a G.P. is to the sum of the first six terms as 125 : 152. Find the common ratio of the G.P.

13. The sum of first three terms of a G.P. is $\frac{39}{10}$ and their product is 1. Find the common ratio and the terms.

14. How many terms of G.P. 3, 3^2, 3^3, … are needed to give the sum 120 ?

15. The sum of first three terms of a G.P. is 16 and the sum of the next three terms is 128. Determine the first term, the common ratio and the sum to n terms of the G.P.

16. Given a G.P. with a = 729 and 7^{th} term 64, determine S_7.

17. Find a G.P. for which sum of the first two terms is –4 and the fifth term is 4 times the third term.

18. Find three numbers a, b, c between 2 and 18 such that (i) their sum is 25 (ii) the numbers 2, a, b are consecutive terms of an A.P. and (iii) the numbers b, c, 18 are consecutive terms of a G.P.

19. Sum to n terms

(i) 8 + 88 + 888 + ……..

(ii) .6 + .66 + .666 + ……..

20. Evaluate $\sum_{k=1}^{11}(2+3^k)$.

21. If the 4^{th}, 10^{th} and 16^{th} terms of a G.P. are x, y and z., respectively. Prove that x, y, z are in G.P.

22. Show that the products of the corresponding terms of the sequences *a*, *ar*, ar^2, …. ar^{n-1} and A, AR, AR^2 ,…. AR^{n-1} form a G.P. and find the common ratio.

23. Find four numbers forming a geometric progression in which the third term is greater than the first term by 9, and the second term is greater than the 4^{th} by 18.

24. If the p^{th}, q^{th}, and r^{th} terms of a G.P. are a, b and c respectively. Prove that $a^{q-r} b^{r-p} c^{p-q} = 1$.

25. If the first and the n^{th} terms of a G.P. are a and b, respectively, and if P is the product of *n* terms, prove that $P^2 = (ab)^n$.

26. Show that the ratio of the sum of first *n* terms of a G.P. to the sum of terms from. $(n+1)^{th}$ to $(2n)^{th}$ term is $\frac{1}{r^n}$.

27. If a, b, c, d and p are different real numbers such that
$(a^2 + b^2 + c^2)p^2 - 2(ab + bc + cd)\, p + (b^2 + c^2 + d^2) \leq 0$, then show that a, b, c and d are in G.P.

28. If p, q, r are in G.P. and the equations, $px^2 + 2qx + r = 0$ and $dx^2 + 2ex + f = 0$ have a common root, then show that $\frac{d}{p}, \frac{e}{q}, \frac{f}{r}$ are in A.P.

29. Find the natural number n for which $\sum_{x=1}^{n} f(x) = 120$, where the function f satisfies the relation f(x + y) = f(x) . f(y) for all natural numbers x, y and further f(1) = 3.

30. The sum of some terms of G.P. is 315 whose first term and the common ratio are 5 and 2, respectively. Find the last term and the number of terms.

31. The first term of a G.P. is 1. The sum of the third term and the fifth term is 90. Find the common ratio of G.P.

32. The sum of three numbers in G.P. is 56. If we subtract 1, 7, 21 from these numbers in that order, we obtain an A.P. Find the numbers.

33. A G.P. consists of an even number of terms. If the sum of all the terms is 5 times the sum of terms occupying odd places, then find its common ratio.

34. If $\frac{a+bx}{a-bx} = \frac{b+cx}{b-cx} = \frac{c+dx}{c-dx}$ $(x \neq 0)$, then show that a, b, c and d are in G.P.

35. If a and b are the roots of $x^2 - 3x + p = 0$ and c, d are roots of $x^2 - 12x + q = 0$, where a, b, c, d form a G.P. Prove that (q + p) : (q – p) = 17 : 15.

36. If a, b, c are in A.P., b, c, d are in G.P. and $\frac{1}{c}, \frac{1}{d}, \frac{1}{e}$ are in A.P. prove that a, c, e and in G.P.

37. The number of bacteria in a certain culture doubles every hour. If there were 30 bacteria present in the culture originally, how many bacteria will be present at the end of 2^{nd} hour, 4^{th} hour and n^{th} hour?

38. What will Rs. 500 amounts to in 10 years after its deposit in a bank which pays annual interest rate of 10% compounded annually?

39. A person writes a letter to four of his friends. He asks each one of the them to copy the letter and mail to four different persons with instruction that they move the chain similarly. Assuming that the chain is not broken and that it costs 50 paise to mail one letter, find the amount spent on the postage when 8^{th} set of letter is mailed.

40. If a, b, c are in G.P. and $a^{\frac{1}{x}} = b^{\frac{1}{y}} = c^{\frac{1}{z}}$, prove that x, y, z are in A.P.

Sum of an infinite G.P.

Theorem : **The sum of an infinite G.P. with first term a and common ratio r(–1 < r < 1 i.e. |r| < 1) is** $S_\infty = \frac{a}{1-r}$.

Proof: Consider an infinite G.P. with first term a and common ratio r, where –1 < r < 1 i.e. |r| <1. The sum of n terms of this G.P. is given by

$$S_n = a\left(\frac{1-r^n}{1-r}\right) = \frac{a}{1-r} - \frac{ar^n}{1-r} \qquad ....(i)$$

Since $-1 < r < 1$, therefore r^n decreases as n increases and r^n tends to zero as n tends to infinity i.e. $r^n \to 0$ as $n \to \infty$.

$\therefore \quad \frac{ar^n}{1-r} \to 0$ as $n \to \infty$. Hence, from (i), the sum of an infinite G.P. is given by

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{a}{1-r} - \frac{ar^n}{1-r}\right) = \frac{a}{1-r} \text{ if } |r| < 1$$

Note : If $r \geq 1$, then the sum of an infinite G.P. tends to infinity.

Illustration 76

Find the sum of an infinite G.P. whose first term is 28 and the fourth term is $\frac{4}{49}$.

Let a be the first term and r the common ratio where $-1 < r < 1$.

Then $a = 28$ and 4th term $= ar^3 = 28r^3$. $\therefore 28r^3 = \frac{4}{49}$, or $r^3 = \frac{4}{49 \times 28}$; $= \frac{1}{7^3}$; $\therefore r = \frac{1}{7}$

$\therefore \quad -1 < r < 1$ and $S_\infty = \frac{a}{1-r} = \frac{28}{1-\frac{1}{7}} = 28 \times \frac{7}{6} = \frac{98}{3}$

Hence the required sum of the infinite G.P. is $\frac{98}{3}$ or $32\frac{2}{3}$.

Illustration 77

(i) Use a Geometric series to express 0.555 .... as a rational number .

(ii) Represent as a common fraction, the decimal $24.28\overline{137}$.

(i) 0.555 ....

$= 0.5 + 0.05 + 0.005 + \ldots$.. to infinity.

This is an infinite G.P. whose first term is 0.5 and common ratio is .1.

Also $-1 < .1 < 1$.

$\therefore$ The sum of the infinite G.P. $= \frac{a}{1-r} = \frac{0.5}{1-.1} = \frac{0.5}{0.9} = \frac{5}{9}$

(ii) $24.28\overline{137} = 24.28137137137 \ldots\ldots$

$= 24.28 + .00137 + .00000137 + .00000000137 + \ldots\ldots.$

$= 24.28 + \frac{137}{10^5} + \frac{137}{10^8} + \frac{137}{10^{11}} + \ldots\ldots \text{ to } \infty$

$= 24.28 + \left(\text{an infinite G.P. with } a = \frac{137}{10^5} \text{ and } r = \frac{1}{10^3}\right)$

$= 24.28 + \frac{137/10^5}{1-1/10^3} = 24.28 + \frac{137}{10^5} \times \frac{10^3}{999} = \frac{2428}{100} + \frac{137}{99900} = \frac{2425709}{99900}$

Illustration 78

If x = 1 + a + a^2 + ... to ∞ and y = 1 + b + b^2 + ... to ∞, prove that 1 + ab + a^2b^2 + ... to ∞ = $\frac{xy}{x+y-1}$, when –1 < a < 1 and –1 < b < 1.

Here $\quad x = 1 + a + a^2 + \dots \text{ to } \infty = \frac{1}{1-a}$ $\qquad [-1 < a < 1]$

and $\quad y = 1 + b + b^2 + \dots \text{ to } \infty = \frac{1}{1-b}$ $\qquad [-1 < b < 1]$

Again, since –1 < a < 1 and –1 < b < 1 ; ∴ –1 < ab < 1.

∴ $\quad 1 + ab + a^2b^2 + \dots \text{ to } \infty = \frac{1}{1-ab}$, where –1 < ab < 1.

and $$\frac{xy}{x+y-1} = \frac{\frac{1}{1-a}\times\frac{1}{1-b}}{\frac{1}{1-a}+\frac{1}{1-b}-1} = \frac{\frac{1}{(1-a)(1-b)}}{\frac{1-b+1-a-(1-a-b+ab)}{(1-a)(1-b)}}$$

$$= \frac{1}{1-b+1-a-1+a+b-ab} = \frac{1}{1-ab}$$

Hence $1 + ab + a^2b^2 + \dots \text{ to } \infty = \frac{xy}{x+y-1}$

Illustration 79

A person is entitled to receive an annual payment which for each year is less by one tenth of what it was for the year before. If the first payment is Rs. 100, show that he cannot receive more than Rs. 1,000 however long he may live.

Here first payment = 100

Second payment = $100 - \frac{1}{10} \times 100$ = Rs. 90 ;

Third payment = $90 - \frac{1}{10} \times 90$ = Rs. 81, and so on.

∴ $\quad$ The annual payment are Rs. 100, Rs. 90, Rs. 81, ...

which are in G.P. with common ratio $\frac{9}{10}$ (<1), and the sum to infinity of this G.P.

= 100 + 90 + 81 + ... to ∞

$$= \frac{a}{1-r} = \frac{100}{1-\frac{9}{10}} = 100 \times \frac{10}{1} = 1000 .$$

Hence the person cannot receive more than Rs. 1,000 however long he may live.

Illustration 80

If A = 1 + r^a + r^{2a} + ... to ∞ and B = 1 + r^b + r^{2b} + ... ∞, prove that

$$r = \left(\frac{A-1}{A}\right)^{1/a} = \left(\frac{B-1}{B}\right)^{1/b}$$

We have, A = 1 + r^a + r^{2a} + ... ∞ and B = 1 + r^b + r^{2b} + ... to ∞

$\Rightarrow \quad A = \frac{1}{1-r^a}$ and $B = \frac{1}{1-r^b}$

$\Rightarrow \quad 1 - r^a = \frac{1}{A}$ and $1 - r^b = \frac{1}{B}$

$\Rightarrow \quad r^a = \left(1-\frac{1}{A}\right)$ and $r^b = \left(1-\frac{1}{B}\right)$

$\Rightarrow \quad r = \left(\frac{A-1}{A}\right)^{1/a}$ and $r = \left(\frac{B-1}{B}\right)^{1/b}$

Hence $r = \left(\frac{A-1}{A}\right)^{1/a} = \left(\frac{B-1}{B}\right)^{1/b}$

Illustration 81

If $x = \sum_{n=0}^{\infty} \cos^{2n}\theta$, $y = \sum_{n=0}^{\infty} \sin^{2n}\phi$, $z = \sum_{n=0}^{\infty} \cos^{2n}\theta \sin^{2n}\phi$, where $0 < \theta, \phi < \pi/2$, then prove that $xz + yz - z = xy$.

We have,

$$x = \sum_{n=0}^{\infty} \cos^{2n}\theta = 1 + \cos^2\theta + \cos^4\theta + ... \infty \Rightarrow x = \frac{1}{1-\cos^2\theta} \Rightarrow \sin^2\theta = \frac{1}{x}$$

$$y = \sum_{n=0}^{\infty} \sin^{2n}\phi = 1 + \sin^2\phi + \sin^4\phi + ... \infty \Rightarrow y = \frac{1}{1-\sin^2\phi} \Rightarrow \cos^2\phi = \frac{1}{y}$$

and $z = \sum_{n=0}^{\infty} \cos^{2n}\theta \sin^{2n}\phi = 1 + \cos^2\theta \sin^2\phi + \cos^4\theta \sin^4\phi + ... \infty$

$\Rightarrow \quad z = \frac{1}{1-\cos^2\theta\sin^2\phi}$

$\Rightarrow \quad z = \frac{1}{1-(1-\sin^2\theta)(1-\cos^2\phi)}$

$$\Rightarrow \quad z = \frac{1}{1-\left(1-\frac{1}{x}\right)\left(1-\frac{1}{y}\right)} \quad \Rightarrow \quad z = \frac{1}{\frac{1}{x}+\frac{1}{y}-\frac{1}{xy}}$$

$$\Rightarrow \quad z = \frac{xy}{x+y-1} \Rightarrow xz + yz - z = xy.$$

Illustration 82

If |x| < 1 and |y| < 1, find the sum to infinity of the series
$(x + y) + (x^2 + xy + y^2) + (x^3 + x^2y + xy^2 + y^3) + ...$

We have ,

$$(x + y) + (x^2 + xy + y^2) + (x^3 + x^2y + xy^2 + y^3) + ... \infty$$

$$= \frac{1}{x-y}[(x^2 - y^2) + (x^3 - y^3) + (x^4 - y^4) + ... \text{ to } \infty]$$

$$\left[\because \frac{x^n - a^n}{x-a} = x^{n-1} + x^{n-2}a^1 + x^{n-3}a^2 + ... + a^{n-1}, n \in N\right]$$

$$= \frac{1}{x-y}[(x^2 + x^3 + x^4 + ... \text{ to } \infty) - (y^2 + y^3 + y^4 + ... \text{ to } \infty)]$$

$$= \frac{1}{x-y}\left[\frac{x^2}{1-x} - \frac{y^2}{1-y}\right] = \frac{1}{x-y}\frac{[x^2 - x^2y - y^2 + y^2x]}{(1-x)(1-y)} = \frac{1}{(x-y)}\frac{[(x^2-y^2) - xy(x-y)]}{(1-x)(1-y)} = \frac{x+y-xy}{(1-x)(1-y)}$$

PRACTICE ASSIGNMENT IV

1. (i) Find the value of $0.\overline{423}$

   (ii) Find the value of $2.12\overline{345}$

2. The sum of an infinite number of terms of a decreasing G.P. is 3 and the sum of squares of its terms to infinity is also 3. Find the series.

3. The first term of a G.P. exceeds the second term by 4 and the sum to infinity is 9. Find the G.P.

4. The sum of infinite number of terms of a G.P. is 6 and the sum of their squares is $\frac{36}{5}$. Find the G.P.

5. Prove that

   (i) $6^{1/2}. 6^{1/4} . 6^{1/8}$ .... $\infty = 6$

   (ii) $9^{1/3} . 9^{1/9} . 9^{1/27}$ ....$\infty = 3$

6. A square is drawn by joining the midpoints of the sides of a square. A third square is drawn inside the second square in the same way and the process is continued indefinitely. If the side of the square is 10 cm, find the sum of the areas of all the squares so formed.

7. After striking a floor, a certain ball rebounds $\left(\frac{4}{5}\right)$th of the height from which it has fallen. Find the total distance that it travels before coming to rest, if it is gently dropped from a height of 120 m.

8. One side of an equilateral triangle is 18 cm. The midpoints of its sides are joined to form another triangle whose midpoints, in turn, are joined to form still another triangle. The process is continued indefinitely. Find the sum of the (i) perimeters of all the triangles (ii) areas of all the triangles.

Geometric Mean (G.M.)

When three quantities are in G.P., the middle one is called the geometric mean (G.M.) of the other two. If a, m, b are in G.P., then m is the G.M. of a and b and we have $\frac{m}{a} = \frac{b}{m}$ or, $m^2 = ab$;

$\therefore \quad m = \pm\sqrt{ab}$

Of the three numbers 2, 6, 18, number 6 is the G.M. of 2 and 18, since 2, 6, 18 are in G.P. and $\sqrt{2\times 18}$ = 6, –6 is also the G.M. of 2 and 18. [$\because$ 2, –6, 18 are also in G.P.]

Extension. If a, m_1, m_2 , …., m_n, b are in G.P., then the intermediate quantities , m_1, m_2, …, m_n are called the geometric means between a and b.

To insert n geometric means between a and b.

If m_1, m_2, …, m_n be the n geometric means which are to be inserted between a and b, then a, m_1, m_2, …, m_n, b are in G.P. of which a is the first and b is the (n + 2)th term.

If r be the C.R., then $b = a.\ r^{(n+2)-1} = ar^{n+1}$.

and we get $r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$

Hence the required geometric means m_1, m_2, …, m_n are $ar, ar^2, ar^3, \ldots, ar^n$.

i.e. , $\quad a\left(\frac{b}{a}\right)^{\frac{1}{n+1}}, a\left(\frac{b}{a}\right)^{\frac{2}{n+1}}, a\left(\frac{b}{a}\right)^{\frac{3}{n+1}}, \ldots, a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$

Theorem: **If n geometric means are inserted between two quantities, then the product of n geometric means is the nth power of the single geometric mean between the two quantities.**

Proof : Let $G_1, G_2, G_3, \ldots, G_n$ be n geometric means between two quantities a and b.

Then, a, $G_1, G_2, \ldots, G_n$, b is a G.P. Let r be the common ratio of this G.P.

Then $r = (b/a)^{\frac{1}{n+1}}$

and , $G_1 = ar$, $G_2 = ar^2$, $G_3 = ar^3$, …., $G_n = ar^n$.

Now, $G_1. G_2. G_3. \ldots. G_n = (ar)(ar^2)(ar^3)\ldots(ar^n)$

$$= a^n r^{1+2+3+\ldots+n}$$

$$= a^n r^{\frac{n(n+1)}{2}} = a^n\left[\left(\frac{b}{a}\right)^{\frac{1}{n+1}}\right]^{\frac{n(n+1)}{2}} = a^n\left(\frac{b}{a}\right)^{n/2} = a^{n/2}\, b^{n/2}$$

$$= (a^{1/2} b^{1/2})^n = (\sqrt{ab})^n = G^n$$

where , G = $\sqrt{ab}$ is the single geometric mean between a and b.

Some Important Properties of Arithmetic and Geometric Means

Prop. I **If A and G are respectively arithmetic and geometric means between two distinct positive numbers a and b, then A > G.**

Proof : We have

$$A = \frac{a+b}{2} \text{ and } G = \sqrt{ab}$$

$$\therefore \quad A - G = \frac{a+b}{2} - \sqrt{ab} = \frac{a+b-2\sqrt{ab}}{2} = \frac{1}{2}[(\sqrt{a} - \sqrt{b})^2] > 0$$

$$\Rightarrow \quad A > G$$

Prop. II. **If A and G are respectively arithmetic and geometric means between two positive quantities a and b, then the quadratic equation having a, b as its roots is $x^2 - 2Ax + G^2 = 0$.**

Proof : We have,

$$A = \frac{a+b}{2} \text{ and } G = \sqrt{ab}$$

The equation having a, b as its roots is

$$x^2 - x(a+b) + ab = 0 \text{ or , } x^2 - 2Ax + G^2 = 0 \qquad \left[\because A = \frac{a+b}{2} \text{ and } G = \sqrt{ab}\right]$$

Prop. III. **If A and G be the A.M. and G.M. between two positive numbers, then the numbers are A ± $\sqrt{A^2 - G^2}$**

Proof : The equation having its roots as the given number is

$$x^2 - 2Ax + G^2 = 0$$

$$\Rightarrow \quad x = \frac{2A \pm \sqrt{4A^2 - 4G^2}}{2} \Rightarrow x = A \pm \sqrt{A^2 - G^2}$$

Illustration 83

Insert 6 geometric means between $\frac{1}{27}$ and 81.

Let $g_1, g_2, g_3, g_4, g_5, g_6$ be the 6 geometric means between $\frac{1}{27}$ and 81. Then

$\frac{1}{27}, g_1, g_2, g_3, g_4, g_5, g_6$, 81 are in G.P.

$\therefore$ a = 1st term = $\frac{1}{27}$ and 8th term = 81.

Let r be the common ratio of the G.P.

Then 8th term = $a(r)^{8-1} = \frac{1}{27} \times r^7$.

$\therefore \quad \frac{1}{27} \times r^7 = 81$, or, $r^7 = 81 \times 27 = 3^7$: $\therefore r = 3$

$\therefore \quad g_1 = \frac{1}{27} \times 3 = \frac{1}{9}, g_2 = \frac{1}{9} \times 3 = \frac{1}{3}, g_3 = \frac{1}{3} \times 3 = 1$

$g_4 = 1 \times 3 = 3$, $g_5 = 3 \times 3 = 9$, $g_6 = 9 \times 3 = 27$

Hence the required geometric means are

$\frac{1}{9}, \frac{1}{3}$, 1, 3, 9, 27.

Illustration 84

The arithmetic mean of two positive numbers is 15 and their geometric mean is 9. Find the numbers.

Let the two positive numbers be a and b

Then $\frac{a+b}{2} = 15$ and $\sqrt{ab} = 9$ $\left[\because \text{A.M.} = \frac{a+b}{2}, \text{G.M.} = \sqrt{ab}\right]$

or $a + b = 30$ .....(1) and $ab = 81$ .....(2)

$\therefore \quad (a - b)^2 = (a + b)^2 - 4ab = (30)^2 - 4 \times 81$

$= 900 - 324 = 576$

$\therefore \quad a - b = \pm 24$, i.e., $a - b = 24$ ....(3) or $a - b = -24$ .....(4)

Solving (1) and (3), a = 27 and b = 3.

Again solving (1) and (4), a = 3 and b = 27 .

Hence the required numbers are 27, 3 or 3, 27.

Illustration 85

Find the value of n so that $\frac{a^{n+1} + b^{n+1}}{a^n + b^n}$ may be the geometric mean between a and b.

$\frac{a^{n+1} + b^{n+1}}{a^n + b^n}$ is the G.M. between a and b

$\Leftrightarrow \quad \frac{a^{n+1} + b^{n+1}}{a^n + b^n} = \sqrt{ab}$

$\Leftrightarrow \quad a^{n+1} + b^{n+1} = a^{\left(n+\frac{1}{2}\right)} b^{1/2} + a^{1/2} b^{\left(n+\frac{1}{2}\right)}$

$\Leftrightarrow \quad a^{n+1} - a^{\left(n+\frac{1}{2}\right)} b^{1/2} = a^{1/2} b^{\left(n+\frac{1}{2}\right)} - b^{n+1}$

$$\Leftrightarrow \quad a^{\left(n+\frac{1}{2}\right)}(a^{1/2}-b^{1/2}) = b^{\left(n+\frac{1}{2}\right)}(a^{1/2}-b^{1/2})$$

$$\Leftrightarrow \quad a^{\left(n+\frac{1}{2}\right)} = b^{\left(n+\frac{1}{2}\right)} \qquad [\because a^{1/2}-b^{1/2} \neq 0, \text{ as } a \neq b]$$

$$\Leftrightarrow \quad \left(\frac{a}{b}\right)^{\left(n+\frac{1}{2}\right)} = 1$$

$$\Leftrightarrow \quad \left(\frac{a}{b}\right)^{\left(n+\frac{1}{2}\right)} = \left(\frac{a}{b}\right)^{0} \Leftrightarrow n + \frac{1}{2} = 0 \Leftrightarrow n = -\frac{1}{2}$$

Illustration 86

Find two positive numbers whose difference is 12 and whose A.M. exceeds the G.M. by 2.

Let the two numbers be a and b . Then,

$$a - b = 12 \qquad ....(i)$$

It is given that

$$AM - GM = 2$$

$$\Rightarrow \quad \frac{a+b}{2} - \sqrt{ab} = 2$$

$$\Rightarrow \quad a + b - 2\sqrt{ab} = 4$$

$$\Rightarrow \quad (\sqrt{a} - \sqrt{b})^2 = 4$$

$$\Rightarrow \quad \sqrt{a} - \sqrt{b} = \pm 2 \qquad ....(ii)$$

Now, $a - b = 12$

$$\Rightarrow \quad (\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = 12$$

$$\Rightarrow \quad (\sqrt{a} + \sqrt{b}) \times (\pm 2) = 12 \qquad ...(iii)$$

$$\Rightarrow \quad \sqrt{a} + \sqrt{b} = \pm 6 \qquad [\text{Using (ii)}]$$

Solving (ii) and (iii), we get $a = 16, b = 4$

Hence, the required numbers are 16 and 4.

Illustration 87

Find two numbers whose arithmetic mean is 34 and the geometric mean is 16.

Let the two numbers be a and b. Then,

$$AM = 34 \Rightarrow \frac{a+b}{2} = 34 \Rightarrow a + b = 68,$$

and $GM = 16 \Rightarrow \sqrt{ab} = 16 \Rightarrow ab = 256$ ....(i)

$$\therefore \quad (a-b)^2 = (a+b)^2 - 4ab \Rightarrow (a-b)^2 = (68)^2 - 4 \times 256 = 3600$$

$$\Rightarrow \quad a - b = 60 \qquad ....(ii)$$

On solving (i) and (ii), we get $a = 64$ and $b = 4$.

Hence , the required numbers are 64 and 4.

Illustration 88

If a, b, c are in G.P. and the equations $ax^2 + 2bx + c = 0$, $b > 0$ and $dx^2 + 2ex + f = 0$, $e > 0$ have a common root, then show that $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in A.P.

Since a, b, c are in G.P. Therefore,

$b^2 = ac$

Now, $ax^2 + 2bx + c = 0$

$\Rightarrow$ $ax^2 + 2\sqrt{ac}\,x + c = 0$

$\Rightarrow$ $(\sqrt{a}x + \sqrt{c})^2 = 0$ $\Rightarrow$ $\sqrt{a}x + \sqrt{c} = 0 \Rightarrow x = -\sqrt{\frac{c}{a}}$

It is given that the equation $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have a common root and the equation $ax^2 + 2bx + c = 0$ has equal roots both equal to $-\sqrt{\frac{c}{a}}$

$\therefore$ $-\sqrt{\frac{c}{a}}$ is a root of the equation $dx^2 + 2ex + f = 0$

$\Rightarrow$ $d\frac{c}{a} - 2e\sqrt{\frac{c}{a}} + f = 0$

$\Rightarrow$ $\frac{d}{a} - 2e\sqrt{\frac{1}{ac}} + \frac{f}{c} = 0$ [Dividing through out by c]

$\Rightarrow$ $\frac{d}{a} - \frac{2e}{b} + \frac{f}{c} = 0$ [$\because$ $b^2 = ac$]

$\Rightarrow$ $2\frac{e}{b} = \frac{d}{a} + \frac{f}{c}$ $\Rightarrow$ $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in A.P.

Illustration 89

If the A.M. and G.M. between two numbers are in the ratio m : n, then prove that the numbers are in the ratio $m + \sqrt{m^2 - n^2} : m - \sqrt{m^2 - n^2}$

Let the two numbers be a and b. Let A and G be respectively the arithmetic and geometric means between a and b.

Then , $A = \frac{a+b}{2}$ and $G = \sqrt{ab}$

$\therefore$ $a + b = 2A$ and $G^2 = ab$ .....(i)

The equation having a, b as its roots is $x^2 - (a + b)x + ab = 0$ or, $x^2 - 2Ax + G^2 = 0$ [Using (i)]

$\therefore \quad x = \dfrac{2A \pm \sqrt{4A^2 - 4G^2}}{2} \Rightarrow x = A \pm \sqrt{A^2 - G^2}$

So, the two numbers are $a = A + \sqrt{A^2 - G^2}$ and $b = A - \sqrt{A^2 - G^2}$

It is given that $A : G = m : n \Rightarrow A = \lambda m$ and $G = \lambda n$ for some λ

Substituting the value of A and G in $a = A + \sqrt{A^2 - G^2}$ and $b = A - \sqrt{A^2 - G^2}$, we get

$$\frac{a}{b} = \frac{\lambda m + \sqrt{\lambda^2 m^2 - \lambda^2 n^2}}{\lambda m - \sqrt{\lambda^2 m^2 - \lambda^2 n^2}} \Rightarrow \frac{a}{b} = \frac{m + \sqrt{m^2 - n^2}}{m - \sqrt{m^2 - n^2}} \Rightarrow \quad a : b = (m + \sqrt{m^2 - n^2}) : (m - \sqrt{m^2 - n^2})$$

Illustration 90

Let x be the arithmetic mean and y, z be two geometric means between any two positive numbers. Then prove that $\dfrac{y^3 + z^3}{xyz} = 2$

Let a and b be two positive numbers. Then,

x = A.M. of a and b $\Rightarrow x = \dfrac{a + b}{2}$ ...(i)

It is given that y and z are two geometric means between a and b. Then a, y, z, b is a

G.P. with common ratio $\left(\dfrac{b}{a}\right)^{\frac{1}{2+1}} = \left(\dfrac{b}{a}\right)^{1/3}$

$\therefore \quad y = ar \Rightarrow y = a\left(\dfrac{b}{a}\right)^{1/3} \Rightarrow y = b^{1/3} a^{2/3}$

$z = ar^2 \Rightarrow z = a\left(\dfrac{b}{a}\right)^{2/3} \Rightarrow z = b^{2/3} a^{1/3}$

$\therefore \quad y^3 + z^3 = (b^{1/3} a^{2/3})^3 + (b^{2/3} a^{1/3})^3 = ba^2 + b^2 a = ab(a + b)$

and $\quad yz = (b^{1/3} a^{2/3})(b^{2/3} a^{1/3}) = ab.$

Thus, $\quad y^3 + z^3 = yz(a + b) \Rightarrow y^3 + z^3 = yz(2x)$ [Using (i)]

$\Rightarrow \quad \dfrac{y^3 + z^3}{xyz} = 2$

Illustration 91

If one geometric mean G and two arithmetic means A_1 and A_2 be inserted between two given quantities, prove that $G^2 = (2A_1 - A_2)(2A_2 - A_1)$

Let a and b be two given quantities. Then , G is the geometric mean of a and b

$\therefore \quad G = \sqrt{ab} \Rightarrow G^2 = ab$

It is given that A_1, A_2 are two arithmetic means between a and b. Then a, A_1, A_2, b is an A.P. with common difference d $= \frac{b-a}{3}$

$\therefore$ $A_1 = a + d = a + \frac{b-a}{3} = \frac{2a+b}{3}$,

$A_2 = a + 2d = a + \frac{2(b-a)}{3} = \frac{a+2b}{3}$

$\Rightarrow$ $2A_1 - A_2 = 2\left(\frac{2a+b}{3}\right) - \left(\frac{a+2b}{3}\right) = a$

and $2A_2 - A_1 = 2\left(\frac{a+2b}{3}\right) - \left(\frac{2a+b}{3}\right) = b$

$\Rightarrow$ $(2A_1 - A_2)(2A_2 - A_1) = ab$

$\Rightarrow$ $(2A_1 - A_2)(2A_2 - A_1) = G^2$.

PRACTICE ASSIGNMENT V

1. Insert 6 geometric means between 27 and $\frac{1}{81}$.

2. Insert 5 geometric means between 16 and $\frac{1}{4}$.

3. Insert 5 geometric means between $\frac{32}{9}$ and $\frac{81}{2}$.

4. Find the geometric means of the following pairs of numbers.

(i) 2 and 8 (ii) a^3b and ab^3 (iii) – 8 and – 2

5. If one A.M., A and two geometric means G_1 and G_2 inserted between any two positive numbers, show that $\frac{G_1^2}{G_2} + \frac{G_2^2}{G_1} = 2A$.

6. If the A.M. of two positive numbers a and b (a > b) is twice their geometric mean. Prove that a : b = $(2+\sqrt{3}):(2-\sqrt{3})$.

7. Find the two numbers whose A.M. is 25 and GM is 20.

8. Prove that the product of *n* geometric means between two quantities is equal to the n^{th} power of a geometric mean of those two quantities.

9. If a is the A.M. of b and c and the two geometric means are G_1 and G_2, then prove that $G_1^3 + G_2^3 = 2abc$.

10. The sum of two numbers is 6 times their geometric mean, show that numbers are in the ratio (3 + $2\sqrt{2}$) : (3 – $2\sqrt{2}$).

11. Find the value of n so that $\frac{a^{n+1} + b^{n+1}}{a^n + b^n}$ may be the geometric mean between a and b.

12. If A and G be the A.M. and G.M. respectively between two positive numbers, prove that the numbers are $A \pm \sqrt{(A+G)(A-G)}$.

13. If A.M. and G.M. of roots of a quadratic equation are 8 and 5 respectively, then obtain the quadratic equation.

SOME SPECIAL SEQUENCES

ARITHMETICO- GEOMETRIC SERIES(A.G.)

A sequence in which each term is the product of the corresponding terms of an A.P. and a G.P., is known as an **Arithmetico-Geometric Sequence.**

A typical A.P. is a, a + d, a + 2d, …….

A typical G.P. is 1, r, r^2, …..

A typical arithmetico-geometric sequence is

a , (a + d) r, (a + 2d) r^2 , ……

The nth term of this sequence is $[a + (n - 1) d] \times r^{n-1}$.

Example 1. **$1 + 2x + 3x^2 + 4x^3 + ...$ is an arithmetico-geometric series whose corresponding A.P. and G.P. are 1, 2, 3, 4, …. and 1, x, x^2, x^3, ….. respectively.**

Example 2. **$1 - 3x + 5x^2 - 7x^3 + ...$ is an arithmetico-geometric series whose corresponding A.P. and G.P. are 1, 3, 5, 7, …. and 1, –x, x^2, $-x^3$, …. respectively.**

To find the sum of n terms of an A.G. series

Let the arithmetico-geometric series be

$$a + (a + d) r + (a + 2d)r^2 + ...+ [a + (n - 1)d] \times r^{n-1}$$

and let S_n be the sum of n terms of this series. Then,

$S_n = a + (a + d)r + (a + 2d)r^2 + ... + [a + (n - 1)d] \times r^{n-1}$

$rS_n = ar + (a + d)r^2 + ... + [a + (n - 2) d] \times r^{n-1} + [a + (n - 1) d] \times r^n$

On subtraction, we obtain

$(1 - r) S_n = a + dr + dr^2 + ... + dr^{n-1} - [a + (n - 1) d] \times r^n$

$= a + dr (1 + r + ... + r^{n-2}) - [a + (n - 1) d] \times r^n$

$= a + dr\left(\frac{1-r^{n-1}}{1-r}\right) - [a + (n - 1) d] \times r^n$ [Summing the G.P.]

$\therefore \quad S_n = \frac{a}{(1-r)} + \frac{dr}{(1-r)^2} - \frac{dr^n}{(1-r)^2} - \frac{[a+(n-1)d]\times r^n}{(1-r)}$ $(r \neq 1)$

Sum of an infinite A.G. Series

Let $|r| < 1$. Then $r^n \to 0$ and $r^{n-1} \to 0$, when $n \to \infty$.

$\therefore \quad S_\infty = \frac{a}{(1-r)} + \frac{dr}{(1-r)^2}$, which is the sum to infinity of the A.G. Series.

Illustration 92

Find the nth term of the series $1+\frac{2}{3}+\frac{3}{3^2}+\frac{4}{3^3}+\ldots$

The given series is an arithmetico-geometric series whose corresponding A.P. and G.P. are

1, 2, 3, 4, …. and $1, \frac{1}{3}, \frac{1}{3^2}, \frac{1}{3^3}, \ldots$ respectively.

nth term of A.P. = $[1 + (n - 1) \times 1] = n$;

nth term of G.P. = $\left[1\times\left(\frac{1}{3}\right)^{n-1}\right] = \frac{1}{3^{n-1}}$ nth term of the given series = $n \cdot \frac{1}{3^{n-1}} = \frac{n}{3^{n-1}}$

Illustration 93

Find the sum to infinity of the series $1+\frac{2}{3}+\frac{3}{3^2}+\frac{4}{3^3}+\ldots\ldots\infty$,

Let S_∞ denote the sum to infinity of the given series. Then,

$$S_\infty = 1+\frac{2}{3}+\frac{3}{3^2}+\frac{4}{3^3}+\ldots\ldots\infty$$

$$\frac{1}{3}S_\infty = \frac{1}{3}+\frac{2}{3^2}+\frac{3}{3^3}+\ldots\ldots\infty$$

On subtraction, we get

$$\frac{2}{3}S_\infty = 1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\ldots\ldots\infty$$

$$= \frac{1}{\left(1-\frac{1}{3}\right)} = \frac{3}{2} \quad \text{[Taking the sum of the infinite G.P.]}$$

$$\therefore \quad S_\infty = \left(\frac{3}{2}\times\frac{3}{2}\right) = \frac{9}{4}$$

Illustration 94

Find the sum of the infinite series $1 - 3x + 5x^2 - 7x^3 + \ldots.\infty$, when $|x| < 1$.

Let S_∞ denote the sum of the given infinite series. Then,

$$S_\infty = 1 + 3(-x) + 5(-x)^2 + 7(-x)^3 + \ldots\ldots\infty$$

$$(-x)S_\infty = (-x) + 3(-x)^2 + 5(-x)^3 + \ldots\infty$$

On subtraction, we get :

$$(1 + x)S_\infty = 1 + 2(-x) + 2(-x)^2 + 2(-x)^3 + \ldots.\infty$$

$$= 1 + 2 \times [(-x) + (-x)^2 + (-x)^3 + \ldots\ldots\infty]$$

$$= 1 + 2 \cdot \frac{(-x)}{[1-(-x)]} \quad \text{[Taking the sum of the infinite G.P.]}$$

$$= \left[1-\frac{2x}{1+x}\right] = \left(\frac{1-x}{1+x}\right)$$

$$\therefore \quad S_\infty = \frac{(1-x)}{(1+x)^2}$$

Illustration 95

If the sum to infinity of the series $3 + 5r + 7r^2 + \ldots\ldots$ is $\frac{44}{9}$, find r.

The given series is an arithmetico-geometric series in which the A.P. is 3, 5, 7 … with a= 3 and d = 2.
And, the G.P. is 1, r, r^2, … with common ratio r.

$\therefore \quad S_\infty = \frac{44}{9}$

$\Rightarrow \quad \frac{a}{1-r} + \frac{dr}{(1-r)^2} = \frac{44}{9}$, where a = 3, d = 2 & r = r.

$\Rightarrow \quad \frac{3}{1-r} + \frac{2r}{(1-r)^2} = \frac{44}{9}$

$\Rightarrow \quad 44r^2 - 79r + 17 = 0$

$\Rightarrow \quad (4r - 1)(11r - 17) = 0$

$\Rightarrow \quad r = \frac{1}{4}$ or $r = \frac{17}{11}$

But $\left|\frac{17}{11}\right| > 1$ & $\left|\frac{1}{4}\right| < 1$, so $r = \frac{1}{4}$ $\qquad (\because |r| < 1)$

Illustration 96

Find the sum of the infinite series $\quad \mathbf{1 + \frac{4}{3} + \frac{9}{3^2} + \frac{16}{3^3} + \ldots\ldots \infty.}$

This is clearly not an arithmetico-goemetric series, since 1, 4, 9, 16, … are not in A.P However, their successive differences (4 – 1), (9 – 4), (16 – 9),…… are in A.P.

$S_\infty = 1 + \frac{4}{3} + \frac{9}{3^2} + \frac{16}{3^3} + \ldots\ldots \infty \qquad \ldots\ldots(1)$

$\frac{1}{3}S_\infty = \frac{1}{3} + \frac{4}{3^2} + \frac{9}{3^3} + \ldots. \infty \qquad \ldots\ldots(2)$

Subtracting (1) & (2),

$\left(1 - \frac{1}{3}\right) S_\infty = 1 + \frac{3}{3} + \frac{5}{3^2} + \frac{7}{3^3} + \ldots\ldots \infty \qquad \ldots\ldots(3)$

$\frac{1}{3}.\frac{2}{3}S_\infty = \frac{1}{3} + \frac{3}{3^2} + \frac{5}{3^3} + \ldots\ldots \infty \qquad \ldots\ldots(4) \qquad$ [multiplying throughout by (1/3)]

On subtracting (4) from (3), we get

$\left(\frac{4}{9}\right). S_\infty = 1 + \frac{2}{3} + \frac{2}{3^2} + \frac{2}{3^3} + \ldots.. \infty$

$= 1 + \frac{2}{3}\left(1 + \frac{1}{3} + \frac{1}{3^2} + \ldots.\infty\right) = 1 + \frac{2}{3} \times \frac{1}{\left(1 - \frac{1}{3}\right)} = 2$, $S_\infty = \frac{9}{2}$

PRACTICE ASSIGNMENT VI

1. Find the sum upto infinity of the series

$1 + 2x + 3x^2 + 4x^3 + \text{-----}$, where $|x| < 1$

2. If the sum to infinity of the series

$1 + (1 + d)\frac{1}{5} + (1 + 2d)\frac{1}{5^2} + (1 + 3d)\frac{1}{5^3} + \ldots\ldots\ldots$ is $\frac{15}{8}$, find d.

3. Find the sum to n terms of the following series

(i) $1 + 4x + 7x^2 + 10x^3 + \ldots\ldots\ldots$

(ii) $1 + \frac{3}{2} + \frac{5}{4} + \frac{7}{8} + \ldots\ldots\ldots$

(iii) $1 - 5 + 9 - 13 + 17 + \ldots\ldots$

4. Find the sum to infinity of the following series

(i) $3 + \frac{5}{4} + \frac{7}{4^2} + \ldots\ldots\infty$

(ii) $\frac{1}{3} + \frac{3}{9} + \frac{5}{27} + \frac{7}{81} + \ldots\ldots\infty$

(iii) $\frac{3}{2} - \frac{5}{6} + \frac{7}{18} - \frac{9}{54} + \ldots\ldots\infty$

5. If the sum to infinity of the series

$3 + (3 + d)\frac{1}{4} + (3 + 2d)\frac{1}{4^2} + \ldots\ldots\ \infty$ is $\frac{44}{9}$, find d.

6. If the sum to infinity of the series

$1 + 2r + 3r^2 + 4r^3 + \ldots$ is 9/4, find r

7. Show that $2^{1/4} \times 4^{1/8} \times 8^{1/16} \times 16^{1/32} \ldots$ to $\infty = 2$

Sum to n terms of some special sequences

I **Sum of first n natural numbers**

The natural numbers 1, 2, 3 …. n form an A.S.

$\therefore$ Their sum = $\frac{n}{2}[2.1 + (n - 1).1] = \frac{1}{2}n(n + 1)$

II **Sum of the squares of first n natural numbers**

Let $S = 1^2 + 2^2 + 3^2 + \ldots + n^2$.

We have $x^3 - (x - 1)^3 \equiv 3x^2 - 3x + 1$

Giving to x values 1, 2, 3, ….n, we have

$$1^3 - 0^3 = 3.1^2 - 3.1 + 1$$
$$2^3 - 1^3 = 3.2^2 - 3.2 + 1$$
$$3^3 - 2^3 = 3.3^2 - 3.3 + 1$$
$$\vdots \qquad \vdots$$
$$n^3 - (n - 1)^3 = 3.n^2 - 3.n + 1$$

Adding, we get

$$n^3 = 3(1^2 + 2^2 + 3^2 + \ldots + n^2) - 3(1 + 2 + 3 + \ldots + n) + n$$

$$\therefore \quad 3S = n^3 + \frac{3}{2}n(n + 1) - n = \frac{1}{2}n[2n^2 + 3(n + 1) - 2]$$

$$= \frac{1}{2}n(2n^2 + 3n + 1) = \frac{1}{2}n(n + 1)(2n + 1)$$

$\therefore \quad S = \frac{1}{6} n(n + 1)(2n + 1)$

III Sum of the cubes of first n natural numbers

Let $S = 1^3 + 2^3 + 3^3 + ... + n^3$

we have $x^4 - (x - 1)^4 \equiv 4x^3 - 6x^2 + 4x - 1$

Giving to x values, 1, 2, 3, ...n, we have

$$1^4 - 0^4 = 4.1^3 - 6.1^2 + 4.1 - 1$$
$$2^4 - 1^4 = 4.2^3 - 6.2^2 + 4.2 - 1$$
$$3^4 - 2^4 = 4.3^3 - 6.3^2 + 4.3 - 1$$

. . .
. . .
. . .
. . .
. . .
. . .

$n^4 - (n - 1)^4 = 4.\ n^3 - 6.n^2 + 4.n - 1$

By adding, we have

$n^4 = 4(1^3 + 2^3 + 3^3 + ... + n^3) - 6(1^2 + 2^2 + 3^2 + ... + n^2) + 4\ (1 + 2 + 3 + ... + n) - n$

or $\quad n^4 = 4S - 6\ .\ \frac{1}{6} n(n + 1)\ (2n + 1) + 4.\ \frac{1}{2} n(n + 1) - n$

$\therefore \quad 4S = n^4 + n + n(n + 1)(2n + 1) - 2n(n + 1)$

$= n(n + 1)(n^2 - n + 1) + n(n + 1)(2n + 1) - 2n(n + 1)$

$= n(n + 1)(n^2 - n + 1 + 2n + 1 - 2) = n(n + 1)(n^2 + n) = n^2\ (n + 1)^2$

$\therefore \quad S = \left[\frac{n(n+1)}{2}\right]^2$

REMARK 1. Sometimes for the sake of convenience the sum of a sequence is also denoted by putting the Greek letter $\sum$ (Sigma) before its general term . For example, 1 + 2 + 3 + ... + n can be written as $\sum n$, $1^2 + 2^2 + ... + n^2$ is denoted by $\sum n^2$ and $1^3 + 2^3 + ... + n^3$ by $\sum n^3$. Thus, we have

$$\sum n = 1 + 2 + ... + n = \frac{n(n+1)}{2}$$

$$\sum n^2 = 1^2 + 2^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum n^3 = 1^3 + 2^3 + ... + n^3 = \left\{\frac{n(n+1)}{2}\right\}^2$$

and, $\quad \sum a = a + a + ... + a = na$

(n terms)

REMARK 2. Proceeding as above, we have

$$\sum n^4 = 1^4 + 2^4 + ... + n^4 = \frac{n(n+1)(6n^3+9n^2+n-1)}{30}$$

To find the sum to n terms of a given series of natural numbers, we may adopt the following algorithm :

ALGORITHM

STEP I Write nth term of the given series.

STEP II Simplify nth term and express it as a polynomial in n i.e. $T_n = an^3 + bn^2 + cn + d$

STEP III Take the summation from 1 to n i.e.

$$\sum_{k=1}^{n} T_k = a\left(\sum_{k=1}^{n} k^3\right) + b\left(\sum_{k=1}^{n} k^2\right) + c\left(\sum_{k=1}^{n} k\right) + \sum_{k=1}^{n} d.$$

STEP IV Use the formulae for $\sum_{k=1}^{n} k$, $\sum_{k=1}^{n} k^2$ and $\sum_{k=1}^{n} k^3$ and obtain the sum.

Illustration 97

Sum up the following series 2.5 + 5.8 + 8.11 + …. to n terms

If T_n denotes the nth term of the given series , then

T_n = (nth term of A.P. 2, 5, 8, ….) × (nth term of A.P. 5, 8, 11, …)

$= [2 + (n - 1) 3][5 + (n - 1) 3] = (3n - 1)(3n + 2) = 9n^2 + 3n - 2.$

Putting n = 1, $T_1 = 9(1^2) + 3(1) - 2$

Putting n = 2, $T_2 = 9(2^2) + 3(2) - 2$

.. … … ….

Putting n = n, $T_n = 9(n^2) + 3(n) - 2$

Adding vertically,

$$S_n = 9(1^2 + 2^2 + ... + n^2) + 3(1 + 2 + ... + n) - 2n$$

$$= 9\left[\frac{n(n+1)(2n+1)}{6}\right] + 3\left[\frac{n(n+1)}{2}\right] - 2n$$

$$= \frac{3}{2} n(n + 1)(2n + 1) + \frac{3}{2} n(n + 1) - 2n$$

$$= \frac{n}{2} [3(n + 1)(2n + 1) + 3(n + 1) - 4]$$

$$= \frac{n}{2} [3(2n^2 + 3n + 1) + 3(n + 1) - 4]$$

$$= \frac{n}{2} [6n^2 + 9n + 3 + 3n + 3 - 4]$$

$$= \frac{n}{2} [6n^2 + 12n + 2]$$

Hence $S_n = n(3n^2 + 6n + 1)$

Illustration 98

Find the sum of the series 1.2.3 + 2.3.4 + 3.4.5 + … to n terms.

t_n = (nth term of 1, 2, 3…) × (nth term of 2, 3, 4…) × (nth term of 3, 4, 5, ….)

$= [1 + (n - 1) \times 1][2 + (n - 1) \times 1][3 + (n - 1) \times 1]$

$= n(n + 1)(n + 2)$

or $t_n = n^3 + 3n^2 + 2n$

$\therefore$ $S_n = \sum n^3 + 3\sum n^2 + 2\sum n$

$$= \frac{n^2(n+1)^2}{4} + \frac{3n(n+1)(2n+1)}{6} + \frac{2n(n+1)}{2} = \frac{n^2(n+1)^2}{4} + \frac{n(n+1)(2n+1)}{2} + n(n+1)$$

$$= \frac{n(n+1)}{4}[n(n + 1) + 2(2n + 1) + 4] = \frac{n(n+1)}{4}(n^2 + n + 4n + 2 + 4)$$

$$= \frac{n(n+1)(n^2+5n+6)}{4} = \frac{n(n+1)(n+2)(n+3)}{4}$$

Illustration 99

Find the sum of n terms of the series whose nth term is

(i) $n(n + 3)$

(ii) $(n^2 + 2^n)$

(i) $S_n = \sum_{k=1}^{n}(k^2+3k)$ [$\because t_n = (n^2 + 3n)$]

$$= \sum_{k=1}^{n}k^2 + 3\left(\sum_{k=1}^{n}k\right)$$

$$= \frac{1}{6}n(n + 1)(2n + 1) + 3.\frac{1}{2}n(n + 1)$$

$$= \frac{n(n+1)}{6}.[(2n + 1) + 9] = \frac{1}{3}n(n + 1)(n + 5)$$

(iii) $t_n = n^2 + 2^n$

Putting n = 1, 2, 3, …., n successively, we get

$t_1 = 1^2 + 2^1$;
$t_2 = 2^2 + 2^2$;
$t_3 = 3^2 + 2^3$;
… … …
$t_n = n^2 + 2^n$.

Adding column wise, we get

$S_n = (t_1 + t_2 + … + t_n) = (1^2 + 2^2 + …. + n^2) + (2 + 2^2 + 2^3 + ….. + 2^n)$

$$= \left(\sum_{k=1}^{n}k^2\right) + \left[\frac{2(2^n-1)}{(2-1)}\right] = \frac{1}{6}n(n + 1)(2n + 1) + 2(2^n - 1)$$

Illustration 100

Sum to n terms of each of the following series :

(i) 1.2 + 2.3 + 3.4 + 4.5 + …………

(ii) $1.2^2 + 2.3^2 + 3.4^2 + 4.5^2$ + …..

(iii) 1.3.5 + 3.5.7 + 5.7.9 + ….

(iv) $3^2 + 5^2 + 7^2 + 9^2$ + ….

(i) nth term of 1, 2, 3, 4, …. = 1 + (n – 1) . 1 = n

nth term of 2, 3, 4, 5, …. = 2 + (n – 1) . 1 = n + 1

∴ T_n = nth term of the given series = n (n + 1)

i.e., $T_n = n^2 + n$ ….(1)

Putting n = 1, 2, 3, …., n and then adding

$T_1 + T_2 + …. T_n = (1^2 + 1) + (2^2 + 2) + (3^2 + 3) + …. + (n^2 + n)$

i.e, $S_n = [1^2 + 2^2 + 3^2 + …. + n^2] + [1 + 2 + 3 + …. + n]$

$= \sum n^2 + \sum n = \dfrac{n(n+1)(2n+1)}{6} + \dfrac{n(n+1)}{2} = \dfrac{1}{3} n(n + 1)(n + 2)$.

(ii) The given series is

$1.2^2 + 2.3^2 + 3.4^2 + 4.5^2 + ...$

nth term of 1, 2, 3, 4, ... = 1 + (n – 1) . 1 = n

nth term of 2, 3, 4, 5, ... = 2 + (n – 1) . 1 = n + 1

∴ T_n = nth term of the given series

$= n. (n + 1)^2$

i.e., $T_n = n[n^2 + 2n + 1]$

i.e., $T_n = n^3 + 2n^2 + n$ .....(2)

Putting n = 1, 2, 3, ..., n in (2) and adding we get

$S_n = \sum n^3 + 2 \sum n^2 + \sum n$

$= \dfrac{n^2(n+1)^2}{4} + \dfrac{2.n(n+1)(2n+1)}{6} + \dfrac{n(n+1)}{2}$

$= \dfrac{n(n+1)(n+2)(3n+5)}{12}$

(iii) The given series is 1.3.5 + 3.5.7 + 5.7.9 + ...

nth term of 1, 3, 5, ... = 1 + (n – 1) 2 = 2n – 1

nth term of 3, 5, 7, ... = 3 + (n – 1). 2 = 2n + 1

nth term of 5, 7, 9, ... =5 + (n – 1). 2 = 2n + 3

∴ T_n = nth term of the gives series

$= (2n-1)(2n + 1) (2n + 3) = (4n^2 - 1)(2n + 3)$

$T_n = 8n^3 + 12n^2 - 2n - 3$ .....(1)

Putting n = 1, 2, 3, 4, ..., n in (1) and adding

$S_n = 8 \sum n^3 + 12 \sum n^2 - 2 \sum n - 3n$

i.e., $S_n = 8 . \dfrac{n^2(n+1)^2}{4} + 12 . \dfrac{n(n+1)(2n+1)}{6} - \dfrac{2n(n+1)}{2} - 3n$

$S_n = n[2n^3 + 8n^2 + 7n - 2]$

(iv) The given series is $3^2 + 5^2 + 7^2 + 9^2 + ...$

nth term of 3, 5, 7, 9, ... = 3 + (n – 1) 2 = 2n + 1

∴ T_n = nth term of given series

i.e. $T_n = (2n + 1)^2$

i.e. $T_n = 4n^2 + 4n + 1$ .....(1)

Putting n = 1, 2, 3, ..., n in (1) and adding, we get

$S_n = 4 \sum n^2 + 4 \sum n + n$

i.e. $S_n = 4 . \dfrac{n(n+1)(2n+1)}{6} + \dfrac{4n(n+1)}{2} + n = \dfrac{1}{3} n (4n^2 + 12n + 11)$

Illustration 101

Show that $\dfrac{1\times 2^2 + 2\times 3^2 + ... + n\times(n+1)^2}{1^2\times 2 + 2^2\times 3 + ... + n^2\times(n+1)} = \dfrac{3n+5}{3n+1}$

The nth term of the series $1 \times 2^2 + 2 \times 3^2 + ... + n \times (n + 1)^2$

$= n \times (n + 1)^2 = n(n^2 + 2n + 1) = n^3 + 2n^2 + n$

∴ $1 \times 2^2 + 2 \times 3^2 + ... + n \times (n + 1)^2$

$= \sum n^3 + 2\sum n^2 + \sum n$

$= \left(\frac{n(n+1)}{2}\right)^2 + 2.\ \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$

$= \frac{n(n+1)}{12}[3n(n+1) + 4(2n+1) + 6] = \frac{n(n+1)}{12}(3n^2 + 11n + 10)$

$= \frac{n(n+1)(n+2)(3n+5)}{12}$ .....(i)

The nth term of the series

$1^2 \times 2 + 2^2 \times 3 + ... + n^2 \times (n+1) = n^2 \times (n+1) = n^3 + n^2$

$\therefore$ $1^2 \times 2 + 2^2 \times 3 + ... + n^2 \times (n+1) = \sum n^3 + \sum n^2$

$= \left(\frac{n(n+1)}{2}\right)^2 + \frac{n(n+1)(2n+1)}{6}$

$= \left(\frac{n(n+1)}{12}\right)[3n(n+1) + 2(2n+1)] = \frac{n(n+1)}{12}(3n^2 + 7n + 2)$

$= \frac{n(n+1)(n+2)(3n+1)}{12}$ ....(ii)

Dividing (i) by (ii), we get

$\frac{1\times 2^2 + 2\times 3^2 + ...+ n\times(n+1)^2}{1^2\times 2 + 2^2\times 3 + ...+ n^2\times(n+1)} = \frac{3n+5}{3n+1}$

Illustration 102

Show that the sum of the cubes of any number of consecutive integers is divisible by the sum of those integers.

Let the consecutive integers be

$n+1, n+2, ..., n+m.$

Now $(n+1)^3 + (n+2)^3 + ... + (n+m)^3$

$= (1^3 + 2^3 + 3^3 + ... + (n+m)^3) - (1^3 + 2^3 + 3^3 + ... + n^3)$

$= \left(\frac{(n+m)(n+m+1)}{2}\right)^2 - \left(\frac{n(n+1)}{2}\right)^2$ ....(i)

and $(n+1) + (n+2) + ... + (n+m)$

$= (1 + 2 + 3 + ... + (n+m)) - (1 + 2 + 3 + ... + n)$

$= \frac{(n+m)(n+m+1)}{2} - \frac{n(n+1)}{2}$ ....(ii)

Dividing (i) by (ii) , we get

$\frac{(n+1)^3 + (n+2)^3 + ... + (n+m)^3}{(n+1) + (n+2) + ... + (n+m)} = \frac{(n+m)(n+m+1)}{2} + \frac{n(n+1)}{2}$

$= \frac{2n^2 + 2mn + 2n + m^2 + m}{2} = n^2 + mn + n + \frac{m(m+1)}{2}$

But for every $m \in N$, $m(m+1)$ is always even integer

$\Rightarrow$ $\frac{m(m+1)}{2}$ is an integer $\Rightarrow$ $n^2 + mn + n + \frac{m(m+1)}{2}$ is an integer.

Hence the result.

Illustration 103

Sum to 20 terms the series $1.3^2 + 2.5^2 + 3.7^2 + ... 20$ terms

t_n =(nth term of 1, 2, 3, ….) × (nth term of 3, 5, 7, …..)^2

$= [1 + (n - 1) \times 1] \times [3 + (n - 1) \times 2]^2$

$= n(3 + 2n - 2)^2 = n(2n + 1)^2$

$t_n = n(4n^2 + 4n + 1) = 4n^3 + 4n^2 + n.$

$\therefore \quad S_n = 4\sum n^3 + 4\sum n^2 + \sum n$

$$= 4 \times \frac{n^2(n+1)^2}{4} + 4 \cdot \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$

$$= \frac{1}{6} n(n + 1)(6n^2 + 14n + 7) \text{ [After simplification]}$$

Now putting n = 20, we get

$$S_{20} = \frac{1}{6}(20)(20 + 1)(6 \times 400 + 14 \times 20 + 7)$$

$$= \frac{1}{6} \times 20 \times 21 \times (2400 + 280 + 7)$$

$$= \frac{1}{6} \times 20 \times 21 \times 2687 = 188090.$$

Illustration 104

Find the sum (S_n) of $1^2 + 3^2 + 5^2 + ...$ to n terms.

$t_n = [1 + (n - 1).\, 2]^2 = (2n - 1)^2 = 4n^2 - 4n + 1$

$\therefore \quad S_n = \sum t_n = \sum(4n^2 - 4n + 1) = 4\sum n^2 - 4\sum n + n$

$$= 4.\, \frac{1}{6} n(n + 1)(2n + 1) - 4 \cdot \frac{1}{2} n(n + 1) + n = \frac{n}{3}(4n^2 - 1)$$

Illustration 105

Find the sum of all possible products of the first n natural numbers taken two by two.

1, 2, 3, ………, n are n natural numbers.

The sum of all the products of first n natural numbers taken two together

$= 1.2 + 1.3 + ... + 1.\, n + 2.3 + 2.4 + ...+ 2.\, n + 3.4 + 3.5 + ... = S$ (say)

We have $(1 + 2 + 3 + ... + n)^2 = (1^2 + 2^2 + 3^2 + ... + n^2) + 2S$

$\therefore \quad 2S = (1 + 2 + 3 + ... + n)^2 - (1^2 + 2^2 + 3^2 + ... + n^2)$

$$= \left(\frac{n(n+1)}{2}\right)^2 - \frac{1}{6} n(n + 1)(2n + 1) = \frac{1}{12} .\, n(n + 1)(3n^2 + 3n - 4n - 2)$$

$$\therefore \quad S = \frac{1}{24} .\, n(n + 1)(3n^2 - n - 2) = \frac{1}{24} .\, n(n + 1)(n - 1)(3n + 2).$$

Illustration 106

Find the sum of 16 terms of the series : $\frac{1^3}{1} + \frac{1^3 + 2^3}{1 + 3} + \frac{1^3 + 2^3 + 3^3}{1 + 3 + 5}$ + …….

$$t_n = \frac{1^3 + 2^3 + 3^3 + ... + n^3}{1 + 3 + 5 + ...\text{to nth term}} = \frac{\sum n^3}{n^2}$$

$$= \frac{\frac{n^2(n+1)^2}{4}}{n^2} = \frac{1}{4}(n+1)^2 = \frac{1}{4}(n^2 + 2n + 1)$$

$$S_n = \frac{1}{4}(\sum n^2 + 2\sum n + n)$$

$$= \frac{1}{4}\left[\frac{n(n+1)(2n+1)}{6} + \frac{2n}{2}(n+1) + n\right]$$

$$= \frac{n}{24}\{(2n^2 + 3n + 1) + 6(n+1) + 6\} = \frac{n}{24}\left[2n^2 + 9n + 13\right]$$

$\therefore \quad S_{16} = \frac{16}{24}(2 \times 256 + 9 \times 16 + 13) = 446.$

Method of Difference

Sometimes the nth term of a series can not be determined by the methods discussed so far. If a series is such that the difference between successive terms are either in A.P. or in G.P., then we determine its nth term by the method of difference and then find the sum of the series by using the formulas for $\sum n$, $\sum n^2$ and $\sum n^3$.

Illustration 107

Find the sum of the following :

(i) $1 + 3 + 6 + 10 + 15 + ...$ to n terms

(ii) $2 + 5 + 10 + 17 + 26 + ...$ to n terms

(i) Let $S_n = 1 + 3 + 6 + 10 + 15 + ... + T_{n-1} + T_n$ .....(1)

and $S_n = 1 + 3 + 6 + 10 + ... + T_{n-1} + T_n$ .....(2)

Subtract (2) from (1), we get,

$0 = 1 + 2 + 3 + 4 + 5 + ... + (T_n - T_{n-1}) - T_n$

$\therefore \quad T_n = 1 + 2 + 3 + 4 + 5 + ...$ n terms

i.e., $T_n = \sum n$

i.e., $T_n = \frac{n(n+1)}{2}$

i.e., $T_n = \frac{1}{2}(n^2 + n)$ ....(3)

Putting n = 1, 2, 3, ..., n in (3) and adding

$S_n = \frac{1}{2}[\sum n^2 + \sum n)$

i.e., $S_n = \frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right]$

i.e., $S_n = \frac{1}{6}n(n+1)(n+2)$

(ii) Let $S_n = 2 + 5 + 10 + 17 + 26 + t_{n-1} + t_n$ ....(1)

and $S_n = 2 + 5 + 10 + 17 + ... + t_{n-1} + t_n.$ ....(2)

Subtract (2) from (1), we get

$0 = 2 + 3 + 5 + 7 + 9 + ... + (t_n - t_{n-1}) - t_n$

i.e., $t_n = 2 + 3 + 5 + 7 + 9 + ...$ n terms i.e., $t_n = 2 + [3 + 5 + 7 + 9 + ... (n-1) \text{ terms}]$

i.e., $t_n = 2 + \left(\dfrac{n-1}{2}\right) [2.3 + (n-2).2]$

i.e., $t_n = 2 + (n-1)(n+1)$ ....(3)

i.e., $t_n = n^2 + 1$

Putting n = 1, 2, 3, ..., n and adding

$$S_n = \sum n^2 + n = \frac{n(n+1)(2n+1)}{6} + n$$

$$= \frac{n}{6}[(n+1)(2n+1) + 6] = \frac{n}{6}[2n^2 + 3n + 7].$$

Illustration 108

Find the sum to n terms of the series

(i) 1 + 3 + 7 + 15 + ...

(ii) 1 + 5 + 14 + 30 + ...

(iii) 1 + 3 + 7 + 13 + ...

(i) Let T_n denotes the nth term and S_n , the sum of first n terms of the given series, then

$S_n = 1 + 3 + 7 + 15 + ... + T_{n-1} + T_n$ ......(1)

Again $S_n = 1 + 3 + 7 + ... +T_{n-1} + T_n$, ...(2)

(1) – (2) gives us

$0 = 1 + 2 + 4 + 8 + ...$ upto n terms $-T_n$

$\Rightarrow T_n = 1 + 2 + 4 + 8 + ...$upto n terms

$$= \frac{1(2^n - 1)}{2-1} = 2^n - 1$$

$\Rightarrow T_1 = 2^1 - 1$

$T_2 = 2^2 - 1$

$T_3 = 2^3 - 1$

....

....

....

$T_n = 2^n - 1$

on adding vertically downwards

$$S_n = (2^1 + 2^2 + 2^3 + ... + 2^n) - n = \frac{2(2^n - 1)}{2-1} - n = 2^{n+1} - 2 - n.$$

(ii) Let T_n denote the nth term and S_n, the sum of first n terms, then

$S_n = 1 + 5 + 14 + 30 + ... + T_{n-1} + T_n$ .....(1)

Again, $S_n = 1 + 5 + 14 + ... +T_{n-1} + T_n$, ...(2)

(1) – (2) gives us

$0 = 1 + 4 + 9 + 16 + ...$ upto n terms $-T_n$

$\Rightarrow T_n = 1^2 + 2^2 + 3^2 + ...$ upto n terms.

$$= \sum n^2 = \frac{n(n+1)(2n+1)}{6} = \frac{1}{6}\left[2n^3 + 3n^2 + n\right]$$

$$\Rightarrow S_n = \frac{1}{6}[2\sum n^3 + 3\sum n^2 + \sum n]$$

$$= \frac{1}{6}\left[\frac{2n^2(n+1)^2}{4} + \frac{3n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right] = \frac{n(n+1)^2(n+2)}{12}$$

(iii) Let T_n denote the nth term and S_n, the sum of first n terms, then

$S_n = 1 + 3 + 7 + 13 + ... + T_{n-1} + T_n$ .....(1)

Again, $S_n = \quad 1 + 3 + 7 + ... \quad + T_{n-1} + T_n$, ...(2)

(1) – (2) gives us

$0 = 1 + 2 + 4 + 6 + ...\text{upto n terms} - T_n$

$\Rightarrow \quad T_n = 1 + (2 + 4 + 6 + ... \text{ upto } (n-1) \text{ terms})$

$$= 1 + \frac{n-1}{2}[2 \times 2 + (n-1-1) \times 2]$$

$$= 1 + \frac{n-1}{2} \times 2n = 1 + n^2 - n = n^2 - n + 1$$

$$\Rightarrow \quad S_n = \sum n^2 - \sum n + n = \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} + n$$

$$= n\left(\frac{2n^2 + 3n + 1 - 3n - 3 + 6}{6}\right) = \frac{n(2n^2+4)}{6} = \frac{2n(n^2+2)}{6} = \frac{n(n^2+2)}{3}$$

Illustration 109

Sum to n terms 3 + 7 + 14 + 24 + 37 + ... n terms

$S_n = 3 + 7 + 14 + 24 + 37 + ... + t_{n-1} + t_n$ ....(i)

and $S_n = 3 + 7 + 14 + 24 + ... \quad + t_{n-1} + t_n$. ....(ii)

Subtracting (ii) from (i), we get

$0 = (3 + 4 + 7 + 10 + 13 + ... \text{ to n terms}) - t_n$

or $t_n = 2 + [1 + 4 + 7 + 10 + 13 + ... \text{ to n terms}]$

$$= 2 + \frac{n}{2}[2.\,1 + (n-1)\,.\,3] = 2 + \frac{n}{2}[2 + 3n - 3]$$

$$= 2 + \frac{n}{2}(3n-1) = \frac{1}{2}(3n^2 - n + 4)$$

$$S_n = \frac{1}{2}[3\sum n^2 - \sum n + \sum 4]$$

$$= \frac{1}{2}\left\{\frac{3n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} + 4n\right\}$$

$$= \frac{n(n+1)}{4}(2n+1-1) + 2n = \frac{n(n+1)}{4}\,.\,2n + 2n = \frac{n^2(n+1)+4n}{2}$$

Illustration 110

Find the nth term and the sum to n term of the series 6 + 9 + 21 + 69 + 261 + ...

Let $S_n = 6 + 9 + 21 + 69 + 261 + ... + T_{n-1} + T_n$ .....(1)

Again $S_n = \quad 6 + 9 + 21 + 69 + ... + T_{n-1} + T_n$, ...(2)

Subtracting (2) from (1),

$0 = 6 + 3 + 12 + 48 + 192 + ... \text{ to n terms} - T_n$

$0 = 6 + (3 + 12 + 48 + 192 + ...\text{to } (n-1) \text{ terms}) - T_n$

$$T_n = 6 + \frac{3(4^{n-1}-1)}{4-1}$$

$= 6 + 4^{n-1} - 1$

$= 5 + 4^{n-1}$

$\therefore \quad S_n = 5\sum 1 + \sum 4^{n-1}$

$= 5(1 + 1 + \ldots\ldots + 1) + (1 + 4 + 4^2 + \ldots\ldots)$

$= 5n + \dfrac{1(4^n - 1)}{4 - 1} = 5n + \dfrac{4^n - 1}{3} = \dfrac{15n + 4^n - 1}{3}$

Illustration 111

Find the sum to n terms of the series $\frac{1}{2} + \frac{3}{4} + \frac{7}{8} + \frac{15}{16} + \ldots\ldots\ldots\ldots$

Let S be the required sum, then

$S = \frac{1}{2} + \frac{3}{4} + \frac{7}{8} + \frac{15}{16} + \ldots\ldots$ to n terms

$= \left(1 - \frac{1}{2}\right) + \left(1 - \frac{1}{4}\right) + \left(1 - \frac{1}{8}\right) + \ldots\ldots$ upto n terms.

$= (1 + 1 + 1 + \ldots.$ upto n terms$) - \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\ldots.\text{to n terms}\right)$

$= n - \dfrac{1}{2}.\dfrac{1 - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}} = n - 1 + \dfrac{1}{2^n}$

PRACTICE ASSIGNMENT VII

1. Find the sum of the series to n terms whose nth term is

(i) $2n^2 - 3n + 5$

(ii) $2n^3 + 3n^2 - 1$

(iii) $n^3 - 3^n$

2. Sum the following series to n terms

(i) 3 + 15 + 35 + 63 + ……

(ii) 1 + 5 + 12 +22 +35 +……

(iii) 5 + 7 + 13 + 31 + 85 + ……

(iv) 3 + 5 + 9 +15 + 23 + ……

3. Find the sum of the following series to n terms

(i) 1.2.5 + 2.3.6 + 3.4.7. + ……

(ii) 1.2.4 + 2.3.7 + 3.4.10+……

(iii) 1 + (1 + 2) + (1 + 2 + 3) + (1 + 2 + 3 + 4) +……

(iv) 3.8 + 6.11 + 9.14 +…….

(v) $1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \ldots\ldots$

(vi) 1.3 + 3.5 + 5.7 + ……

(vii) 1.3.5 + 2.4.6 + 3.5.7 +……

(viii) $1 + \left(1 + \frac{1}{2}\right) + \left(1 + \frac{1}{2} + \frac{1}{4}\right) + \ldots\ldots$

(ix) $1 + \left(1 + \frac{1}{3}\right) + \left(1 + \frac{1}{3} + \frac{1}{9}\right) + \ldots\ldots\ldots$

(x) $1 + \frac{1+2}{2} + \frac{1+2+3}{3} + \ldots\ldots\ldots\ldots$

(xi) $1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \ldots\ldots\ldots$

(xii) $\frac{1}{2\times 5} + \frac{1}{5\times 8} + \frac{1}{8\times 11} + \ldots\ldots$

4. Find the sum of the series

5.6 + 6.7 + 7.8 + ……to 25 terms

5. Find the sum of the series

$2^2 + 5^2 + 8^2 + \ldots\ldots$ to 15 terms

6. Find the sum to n terms the series

$\frac{1^3}{1} + \frac{1^3+2^3}{2} + \frac{1^3+2^3+3^3}{3} + \ldots\ldots$

7. Find the sum of the series

$10^3 + 11^3 + 12^3 + \ldots\ldots + 20^3$

8. If S_1, S_2, S_3 are the sums of first n natural numbers, their squares, their cubes respectively, show that $9S_2^2 = S_3(1 + 8S_1)$.

HARMONIC PROGRESSION

A series of quantities $x_1, x_2, x_3, \ldots, x_n$ is said to be in Harmonic Progression (or simply in H.P.) when their reciprocal $\frac{1}{x_1}, \frac{1}{x_2}, \frac{1}{x_3}, \ldots, \frac{1}{x_n}$ are in Arithmetic Progression. Conversely, if the quantities are in A.P. their reciprocals are in H.P.

$\therefore$ If a, b, c are in H.P. , then $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P.

Another definition. Three quantities a, b, c are said to be in H.P. if a : c = a – b : b – c. Any number of quantities are said to be in H.P. when every three consecutive terms are in H.P.

Note 1. The above two definitions are identical, since one can be easily deduced from the other.

Note 2. Three is no general formula for the sum of any number of quantities in H.P. All questions in H.P. are generally solved by inverting the terms, and making use of the properties of the corresponding A.P.

Harmonic Mean (H.M.)

When three quantities are in H.P, the middle one is called the Harmonic Mean (H.M.) between the other two.

If a, H, b are in H.P., then H is the H.M. of a, b and $\frac{1}{a}, \frac{1}{H}, \frac{1}{b}$ are in A.P.

$\therefore \quad \frac{1}{H} - \frac{1}{a} = \frac{1}{b} - \frac{1}{H}$, or, $\frac{2}{H} = \frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab}$

or, $(a + b) H = 2ab$, or, $H = \frac{2ab}{a+b}$

If $a_1, a_2, a_3 \ldots, a_n$ are n non zero numbers, then the H.M. 'H' of these numbers is given by

$$\frac{1}{H} = \frac{\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}}{n}$$

Illustration 112

If the 7th and 10th terms of an H.P. are respectively $\frac{2}{5}$ and $\frac{2}{7}$, find the 1st and the nth terms of the H.P.

Let the first term and the common difference of the corresponding A.P. be respectively a_1 and d.

Clearly the 7th and the 10th terms of the A.P. are $\frac{5}{2}$ and $\frac{7}{2}$

$\therefore$ $a_1 + 6d = \frac{5}{2}$ ....(1)

and $a_1 + 9d = \frac{7}{2}$ ....(2)

Subtracting (1) from (2), $3d = \frac{7}{2} - \frac{5}{2} = 1$;

$\therefore$ $d = \frac{1}{3}$

From (1), $a_1 + 6 \cdot \frac{1}{3} = \frac{5}{2}$, or, $a_1 = \frac{5}{2} - 2 = \frac{1}{2}$

$\therefore$ nth term of the A.P. $= a_1 + (n - 1) d = \frac{1}{2} + (n - 1) \cdot \frac{1}{3} = \frac{2n+1}{6}$

Hence the first term of H.P. = 2 and the nth term $= \frac{6}{2n+1}$

Illustration 113

Insert 4 harmonic means between 22 and 2.

Let h_1, h_2, h_3, h_4 be the 4 harmonic means between 22 and 2. Then $22, h_1, h_2, h_3, h_4, 2$ are in H.P.

$\therefore$ $\frac{1}{22}, \frac{1}{h_1}, \frac{1}{h_2}, \frac{1}{h_3}, \frac{1}{h_4}, \frac{1}{2}$ are in A.P.

Let d be the common difference of the A.P.

Then $\frac{1}{2} = 6^{th}$ term of the A.P. $= \frac{1}{22} + 5d$,

or $\frac{1}{2} - \frac{1}{22} = 5d$, or, $5d = \frac{10}{22} = \frac{5}{11}$; $\therefore d = \frac{1}{11}$

$\therefore$ $\frac{1}{h_1} = \frac{1}{22} + \frac{1}{11} = \frac{3}{22}, \frac{1}{h_2} = \frac{3}{22} + \frac{1}{11} = \frac{5}{22}$

$\frac{1}{h_3} = \frac{7}{22}, \frac{1}{h_4} = \frac{9}{22}$

Hence $h_1 = \frac{22}{3}$, $h_2 = \frac{22}{5}$, $h_3 = \frac{22}{7}$, $h_4 = \frac{22}{9}$

Illustration 114

Prove that for any two real quantities, A.M. ≥ G.M. ≥ H.M.

Let A, G, H be the Arithmetic, Geometric and Harmonic means between a and b. Then, we have

$$A = \frac{a+b}{2}, G = \sqrt{ab}, H = \frac{2ab}{a+b}$$

$\therefore$ $A.H = \frac{a+b}{2} \times \frac{2ab}{a+b} = ab = G^2.$

Again

$$A - G = \frac{a+b}{2} - \sqrt{ab} = \frac{a+b-2\sqrt{ab}}{2} = \frac{(\sqrt{a})^2 + (\sqrt{b})^2 - 2\sqrt{ab}}{2}$$

$$= \frac{(\sqrt{a}-\sqrt{b})^2}{2} \geq 0 \qquad [\because (\sqrt{a}-\sqrt{b})^2 \geq 0]$$

$\therefore$ $A \geq G$, equality occurring only when a = b.

Again since $A.H = G^2 = G \times G$ and $A \geq G$; $\therefore$ $H \leq G$ i.e., $G \geq H$.

Combining $A \geq G$ and $G \geq H$, we get $A \geq G \geq H$.

Illustration 115

If p, q, r are in A.P. , q, r, s are in G.P. and r, s, t in H.P. show that p, r, t are in G.P.

We have $p + r = 2q$ ....(1)

$qs = r^2$ .....(2)

and $\frac{1}{r} + \frac{1}{t} = \frac{2}{s}$ .....(3)

From (3), $\frac{t+r}{rt} = \frac{2}{s}$, or, $\frac{1}{s} = \frac{t+r}{2rt}$

From (2), $q = \frac{r^2}{s} = r^2 \times \frac{(t+r)}{2rt} = \frac{tr+r^2}{2t}$

$\therefore$ From (1), $p + r = 2 \times \frac{(tr+r^2)}{2t} = r + \frac{r^2}{t}$, or, $p = \frac{r^2}{t}$

or $pt = r^2$. This shows that p, r, t are in G.P.

Illustration 116

The H.M. of two numbers is 4, their A.M. (A) and G.M. (G) satisfy the relation $2A + G^2 = 27$. Find the two numbers.

Let the two numbers be x and y. Then

$$\frac{2xy}{x+y} = \text{H.M.} = 4, \quad \frac{x+y}{2} = A \text{ and } \sqrt{xy} = G.$$

i.e., $xy = 2(x + y)$

$x + y = 2A$ and $xy = G^2$. ....(1)

Now, $2A + G^2 = 27$, or, $x + y + xy = 27$, or, $(x + y) + 2(x + y) = 27$ [By (1)]

or, $3(x + y) = 27$, or, $x + y = 9$ ...(2)

$\therefore$ From (1), $xy = 18$, or, $y = \frac{18}{x}$

From (2), $x + \frac{18}{x} = 9$, or, $x^2 - 9x + 18 = 0$,

or, $(x - 3)(x - 6) = 0$; $\therefore x = 3, 6$.

If $x = 3$, from (2), $y = 9 - 3 = 6$ and if $x = 6$, $y = 3$.

Hence the two numbers are 3 and 6.

Illustration 117

If a, x, y, z, b are in A.P. , x + y + z = 15 and if a, x, y, z, b are in H.P. , $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{5}{3}$, find a and b.

a, x, y, z, b are in A.P. ; let d be the common difference. Then b = 5th term of the A.P.
$= a + 4d$.

or $d = \frac{b-a}{4}$, $\therefore x = a + \frac{b-a}{4}$, $y = a + 2\cdot\left(\frac{b-a}{4}\right)$,

$z = a + 3\left(\frac{b-a}{4}\right)$. Now $x + y + z = 15$.

or $a + \frac{b-a}{4} + a + 2\left(\frac{b-a}{4}\right) + a + 3\left(\frac{b-a}{4}\right) = 15$,

or $\frac{12a + 6b - 6a}{4} = 15$, or, $a + b = 10$ ...(1)

Since a, x, y, z, b are in H.P. ; $\therefore \frac{1}{a}, \frac{1}{x}, \frac{1}{y}, \frac{1}{z}, \frac{1}{b}$ are in A.P. Let d′ be the common difference.

Then $\frac{1}{b}$ = 5th term $= \frac{1}{a} + 4d'$, or, $d' = \frac{a-b}{4ab}$

$$\therefore \quad \frac{1}{x} = \frac{1}{a} + \frac{a-b}{4ab} = \frac{4b + a - b}{4ab} = \frac{a + 3b}{4ab}$$

$$\frac{1}{y} = \frac{1}{a} + 2\left(\frac{a-b}{4ab}\right) = \frac{2a + 2b}{4ab}$$

$$\frac{1}{z} = \frac{1}{a} + 3\left(\frac{a-b}{4ab}\right) = \frac{3a + b}{4ab}$$

Now $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{5}{3}$, or, $\frac{a+3b}{4ab} + \frac{2a+2b}{4ab} + \frac{3a+b}{4ab} = \frac{5}{3}$

or $9(a + b) = 10ab$, or, $ab = 9$

Solving (1) and (2), we get $a = 1, b = 9$ or $a = 9, b = 1$.

Practice Assignment VIII

1. Find the 20th term of the H.P. : 1, $\frac{1}{4}, \frac{1}{7}, \frac{1}{10}$, ...

2. If the 4th and the 7th terms of an H.P. are $\frac{2}{3}$ and $\frac{2}{5}$ respectively, find the first and the nth terms of the H.P.

3. (i) Insert 4 harmonic means between 20 and 5.

(ii) Insert 6 harmonic means between 3 and $\frac{6}{23}$

4. If the pth, qth, rth terms of an H.P. be a, b, c, respectively, prove that $(q - r)\,bc + (r - p)\,ca + (p - q)\,ab = 0$

5. Prove that a, b, c are in A.P., G.P., or H.P. according as $\frac{a-b}{b-c} = \frac{a}{a}, \frac{a}{b}$ or $\frac{a}{c}$ respectively.

6. If a^2, b^2, c^2 are in A.P. , prove that b + c, c + a, a + b are in H.P.

7. If b is the H.M. between a and c, prove that

$$\frac{1}{b-a} + \frac{1}{b-c} = \frac{1}{a} + \frac{1}{c}$$

8. If a, b, c are in A.P. ; b, c, d are in G.P. ; and c, d, e are in H.P. ; prove that $c^2 = ae$.

9. If a, b, c are in G.P. ; $a^x = b^y = c^z$, prove that x, y, z are in H.P.

10. Find two positive numbers such that their A.M. is 10 and G.M. is 8. Also find their H.M.

11. If A be the A.M. and H the H.M. between two numbers a and b, prove that

$$\frac{a-A}{a-H} \times \frac{b-A}{b-H} = \frac{A}{H}$$

12. Rana travels from Delhi to Chandigarh with a uniform speed of u km/hr and returns from Chandigarh to Delhi with a uniform speed of v km/hr. Show that his average speed is the harmonic mean between u and v.

Objective Assignment

1. The second term of an A.P. is (x – y) and the 5th term is (x+y), then its first term is:

(a) $x - \frac{1}{3}y$
(b) $x - \frac{2}{3}y$
(c) $x - \frac{4}{3}y$
(d) $x - \frac{5}{3}y$

2. If 7th and 13th term of an A.P. be 34 and 64 respectively, then its 18th term is:

(a) 87
(b) 88
(c) 89
(d) 90

3. If $\frac{1}{p+q}$, $\frac{1}{q+r}$ and $\frac{1}{r+p}$ are in A.P. then:

(a) p, q, r in A.P.
(b) q^2, p^2, r^2 are in A.P.
(c) p^2, q^2, r^2 are in A.P.
(d) p, q, r in G.P.

4. The maximum sum of the series

$20 + 19\frac{1}{3} + 18\frac{2}{3} + 18 + ...$ is:

(a) 310
(b) 290
(c) 320
(d) none of these

5. The four numbers in A.P., such that their sum is 50 and greatest of them is 4 times the least, are :

(a) 7, 11, 15, 19
(b) 6, 11, 16, 21
(c) 5, 10, 15, 20
(d) none of these

6. The sum of all two digit numbers which when divided by 4, yield unity as remainder, is:

(a) 1100
(b) 1200
(c) 1210
(d) none of these

7. If the sum of n terms of an A.P. is $3n^2 - n$ and its common difference is 6, then its first term is

(a) 2
(b) 3
(c) 1
(d) 4

8. If S_n denotes the sum of n terms of an A.P., then $S_{n+3} - 3S_{n+2} + 3S_{n+1} - S_n$ =

(a) 0
(b) 1
(c) $\frac{1}{2}$
(d) 2

9. A polygon has 25 sides, the lengths of which starting from the smallest side are in A.P. If the perimeter of the polygon is 2100 cm and the length of the largest side 20 times that of the smallest, then the length of the smallest side and the common difference of the A.P. is:

(a) 8 cm, $6\frac{1}{3}$ cm
(b) 6 cm, $6\frac{1}{3}$ cm
(c) 8 cm, $5\frac{1}{3}$ cm
(d) none of these

10. The largest term common to the sequences 1, 11, 21, 31, ...to 100 terms and 31, 36, 41, 46,...to 100 terms is

(a) 381
(b) 471
(c) 281
(d) none of these

11. If x, 2y, 3z are in A.P., where the distinct numbers x, y, z are in G.P., then the common ratio of the G.P. is

(a) 3
(b) $\frac{1}{3}$

(c) 2 **(d)** $\frac{1}{2}$

12. If $S_n = \sum_{n=1}^{n} \frac{1+2+2^2+\ldots\text{to n terms}}{2^n}$, then S_n is equal to

(a) $2^n - (n+1)$ **(b)** $1 - \frac{1}{2^n}$

(c) $n - 1 + \frac{1}{2^n}$ **(d)** $2^n - 1$

13. Three numbers are in G.P., whose sum is 70, if the extremum be each multiplied by 4 and the mean by 5, they will be in A.P. The numbers are

(a) 10, 20, 40 **(b)** 10, 30, 40

(c) 20, 30, 40 **(d)** none of these

14. Three numbers are in G.P. If we double the middle term, we get an A.P. Then the common ratio of the G.P. equals :

(a) $2 \pm \sqrt{3}$ **(b)** $3 \pm \sqrt{2}$

(c) $3 \pm \sqrt{5}$ **(d)** none of these

15. What will be quadratic equation in x when the roots have arithmetic mean A and the geometric mean G?

(a) $x^2 + 2Ax + G^2 = 0$ **(b)** $x^2 + G^2x + A = 0$

(c) $x^2 - 2Ax + G^2 = 0$ **(d)** $x^2 + Gx + A = 0$

16. The sum of the series 0.4 + 0.004 + 0.00004 +.....to ∞ is:

(a) $\frac{11}{25}$ **(b)** $\frac{41}{100}$

(c) $\frac{40}{99}$ **(d)** $\frac{2}{5}$

17. The nth term of a G.P. is 128 and the sum to its n terms is 255. If its common ratio is 2, then its first term is :

(a) 1 **(b)** 3

(c) 8 **(d)** none of these

18. The sum of $(x+2)^{n-1} + (x+2)^{n-2}(x+1) + (x+2)^{n-3}(x+1)^2 + \ldots + (x+1)^{n-1}$ is equal to

(a) $(x+2)^{n-2} - (x+1)^n$ **(b)** $(x+2)^{n-1} - (x+1)^{n-1}$

(c) $(x+2)^n - (x+1)^n$ **(d)** none of these

19. Sum of all terms of an infinite G.P. is $\frac{1}{5}$ times the sum of odd terms. The common ratio is

(a) 2 **(b)** 3

(c) $-\frac{4}{5}$ **(d)** 5

20. The sum of n terms of the series $1 + \frac{1}{2} + \frac{1}{2^2} + \ldots$ is less than 1.999. The greatest value of n is

(a) 10 **(b)** 11

(c) 9 **(d)** none of these

21. If $x = a + a/r + a/r^2 + \ldots\ldots\ldots\infty$

$y = b - b/r + b/r^2 - \ldots\ldots\ldots\infty$

$z = c + c/r^2 + c/r^4 + \ldots\ldots\ldots\infty$

then the value of xy/z is equal to

(a) $\frac{ab}{c}$ **(b)** $\frac{bc}{a}$

(c) $\frac{ca}{b}$ **(d)** none of these

22. If $S = \frac{2}{3} + \frac{8}{9} + \frac{26}{27} + \frac{30}{81} + \ldots\ldots$ to n terms, then the value of S is equal to

(a) $n\left(1 - \frac{1}{3^n}\right)$ **(b)** $1 - \frac{1}{3^n}$

(c) $2 - \frac{1}{2}\left(\frac{2}{3}\right)^n$ **(d)** $n - \frac{1}{2}\left(1 - \frac{1}{3^n}\right)$

23. The value of $(0.2)^{\log_{\sqrt{5}}\left(\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots\ldots\right)}$ is

(a) 4 **(b)** log 4

(c) log 2 **(d)** none of these

24. If a, b, c are three unequal numbers such that a, b, c are in A.P. and b–a, c–b, a are in G.P., then a : b: c is

(a) 1 : 2: 3 **(b)** 1 : 3 : 5

(c) 2 : 3: 4 **(d)** 1 : 2: 4

25. If three positive real numbers a, b, c are in A.P. such that abc = 4, then the minimum possible value of b is

(a) $2^{3/2}$ **(b)** $2^{2/3}$

(c) $2^{1/3}$ **(d)** $2^{5/2}$

26. Sum of the series 1 + 2 + 4 + 7 + ….. + 67 is equal to

(a) 150 **(b)** 230

(c) 298 **(d)** 340

27. The sum of the series

$\frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9} + \ldots$ upto ∞ is

(a) 1/6 **(b)** 1/3

(c) 1/2 **(d)** 5/6

28. The sum of 0.2 + 0.004 + 0.00006 + …. ∞ is

(a) $\frac{200}{891}$ **(b)** $\frac{2000}{9801}$

(c) $\frac{1000}{9801}$ **(d)** none of these

29. Sum of the series 1 + 3 +7 +15 +31 +… to n terms is

(a) $2^n - 2 - n$ **(b)** $2^{n+1} + 2 + n$

(c) $2^{n+1} - 2 - n$ **(d)** none of these

30. Sum of the series

$1 + 2.2 + 3.2^2 + 4.2^3 + \ldots + 100.2^{99}$ is

(a) $100.2^{100} + 1$ **(b)** $99.2^{100} + 1$

(c) $99.2^{99} - 1$ **(d)** $100.2^{100} - 1$

31. Given that n arithmetic means are inserted between two sets of numbers a, 2b and 2a, b,where a, b $\in$ R. Suppose further that mth mean between these two sets of numbers is same, then ratio a : b equals

(a) n – m + 1 : m **(b)** n – m + 1 : n

(c) m : n – m + 1 **(d)** n : n – m + 1

32. If t_n denotes the nth term of the series 2 + 3+ 6+ 11 +18 + ……., then t_{50} is

(a) $49^2 - 1$ **(b)** 49^2

(c) $50^2 + 1$ **(d)** $49^2 + 2$

33. The sum of the products of the ten numbers $\pm 1, \pm 2, \pm 3, \pm 4, \pm 5$ taking two at a time is

(a) 165 **(b)** – 55

(c) 55 **(d)** none of these

34. It is known that $\sum_{r=1}^{\infty} \frac{1}{(2r-1)^2} = \frac{\pi^2}{8}$, then $\sum_{r=1}^{\infty} \frac{1}{r^2}$ is equal to

(a) $\frac{\pi^2}{24}$ **(b)** $\frac{\pi^2}{3}$

(c) $\frac{\pi^2}{6}$ **(d)** none of these

35. The next term of the sequence 1, 5, 14, 30, 55, ...is

(a) 95 **(b)** 90

(c) 85 **(d)** 91

36. The age of Rahul is twice that of the Ramit added to four times that of the Rohit. If the geometric mean of the ages of Ramit and Rohit is $4\sqrt{3}$ and their harmonic mean is 6, then Rahul's age is (assuming that Ramit is elder than Rohit)

(a) 35 **(b)** 40

(c) 45 **(d)** 50

37. If the first two terms of an HP be 2/5 and 12/23 then the largest positive term of the progression is the

(a) 6th term **(b)** 7th term

(c) 5th term **(d)** 8th term

38. If k arithmetic means $u_1, u_2, u_3, \ldots, u_k$ are inserted between 40 and 800 and k harmonic means $v_1, v_2, v_3 \ldots\ldots, v_k$ are inserted between the same two numbers, then $u_3 v_{k-2}$ is equal to

(a) 4000 **(b)** $\frac{32000}{k}$

(c) 32000 **(d)** $\frac{8400}{k}$

39. If $a, b, c \in R^+$ then $\frac{bc}{b+c} + \frac{ac}{a+c} + \frac{ab}{a+b}$ is always

(a) $\le \frac{1}{2}(a + b + c)$ **(b)** $\ge \frac{1}{3}\sqrt{abc}$

(c) $\le \frac{1}{3}(a + b + c)$ **(d)** $\ge \frac{1}{2}\sqrt{abc}$

40. $\log_3 2, \log_6 2, \log_{12} 2$ are in :

(a) A.P. **(b)** G.P.

(c) H.P. **(d)** none of these.

SEQUENCE & SERIES
ANSWERS

Practice Assignment – I

1. (i) 65, 93 (ii) $\frac{49}{128}$ (iii) 729 (iv) $\frac{360}{23}$

2. (i) 3, 11, 35, 107, 323; 3 + 11 + 35 + 107 + 323 + ...

(ii) $-1, \frac{-1}{2}, \frac{-1}{6}, \frac{-1}{24}, \frac{-1}{120}; -1 + \left(\frac{-1}{2}\right) + \left(\frac{-1}{6}\right) + \left(\frac{-1}{24}\right) + \left(\frac{-1}{120}\right) + \ldots\ldots\ldots\ldots$

(iii)2, 2, 1, 0, –1; 2 + 2 + 1 + 0 + (–1) + ...

(iv) 1, 2, $\frac{3}{2}, \frac{5}{3}$ and $\frac{8}{5}$

3. 33
4. 29
5. ....
6. 14 th
7. .....
8. 28, 70, 98
9. A = 5.8, B = 4.4
10. 5
11. 101
12. 10
13. 8
14. (i) 1002001 (ii) 2754000
15. 37570
16. ....
17. (ii) 459 / 676
18. (i) 7 : 16 (ii) 179 : 321
19. .....
20. 5 or 20
21. ......
22. $\frac{n}{2}(5n + 7)$
23. 2q
24. $\frac{179}{321}$
25. n = 102, 331
26. Rs 245
27. 9
28. 300
29. 900
30. 27
31. Rs. 7900000
32. 4

Practice Assignment – II

1. 1, 7, 13
2. 5, 10, 15, 20
3. 75°, 85°, 95°, 105°
4. 2, 4, 6, 8 or 8, 6, 4, 2

5. 2, 4, 6 or 6, 4, 2
6. …………
7. …………
8. …………..
9. ………..
10. 7, 10, 13, 16
11. $\frac{31}{8}, \frac{23}{4}, \frac{61}{8}, \frac{19}{2}, \frac{91}{8}, \frac{53}{4}, \frac{121}{8}$
12. 1
13. 6
14. m = 14
15. 852
16. 20 months
17. Rs. 16680
18. Rs. 39100
19. 25 days
20. ………

Practice Assignment – III

1. $\frac{5}{2^{20}}, \frac{5}{2^n}$
2. 3072
3. ……….
4. –2187
5. $\frac{1}{2}, -\frac{1}{2}$
6. (i) 13th (ii) 12th (iii) 9th
7. ± 1
8. 5
9. ………
10. ………..
11. (i) $\frac{1}{6}[1-(0.1)^{20}]$ (ii) $\frac{\sqrt{7}}{2}(\sqrt{3}+1)\left(3^{\frac{n}{2}}-1\right)$ (iii) $\frac{[1-(-a)^n]}{1+a}$ (iv) $\frac{x^3(1-x^{2n})}{1-x^2}$
12. 3/5
13. $r = \frac{5}{2}$ or $\frac{2}{5}$; Terms are $\frac{2}{5}, 1, \frac{5}{2}$ or $\frac{5}{2}, 1, \frac{2}{5}$
14. 4
15. $\frac{16}{7}; 2; \frac{16}{7}(2^n - 1)$
16. 2059
17. $\frac{-4}{3}, \frac{-8}{3}, \frac{-16}{3}$, …….or 4, – 8, 16, – 32, 64 , ……

18. 5, 8, 12

19. (i) $\frac{8}{81}[10^{n+1} - 10 - 9n]$

(ii) $\frac{2}{27}[9n - 1 + \frac{1}{10^n}]$

20. $22 + \frac{3}{2}(3^{11} - 1)$

21. ....

22. rR

23. 3, –6, 12, –24

24. ....

25. ....

26. ....

27. ....

28. ......

29. 4

30. 160, 6

31. ± 3

32. 8, 16, 32

33. 4

34. .....

35. .....

36. ....

37. 120, 480, 30 (2^n)

38. Rs 500 $(1.1)^{10}$

39. Rs. 43690

40. .....

Practice Assignment – IV

1. (i) $\frac{423}{999}$

(ii) $\frac{70711}{33300}$

2. $\frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \frac{3}{16} + \ldots.$

3. 6, 2, $\frac{2}{3}$

4. 2, $\frac{4}{3}$, $\frac{8}{9}$, ...

5. ......

6. 200 cm^2

7. 1080 m

8. 108 cm, $108\sqrt{3}$ cm^2

Practice Assignment – V

1. $9, 3, 1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}$

2. $8, 4, 2, 1, \frac{1}{2}$

3. $\frac{16}{3}, 8, 12, 18, 27$

4. (i) 4 (ii) a^2b^2 (iii) –4

5. ......

6. ....

7. 40, 10

8. ....

9. ....

10. ....

11. $\frac{-1}{2}$

12. .....

13. $x^2 - 16x + 25 = 0$

Practice Assignment – VI

1. $\frac{1}{(1-x)^2}$

2. 2

3. (i) $\frac{1}{1-x} + \frac{3x(1-x^{n-1})}{(1-x)^2} - \frac{(3n-2)x^n}{1-x}$

   (ii) $6 - \frac{1}{2^{n-3}}\left\{1 + \frac{n}{2} - \frac{1}{2^2}\right\}$

   (iii) $\frac{1}{2}\left[-1 + (4n-1)(-1)^{n-1}\right]$

4. (a) 44/9 (b) 1 (c) 15/16

5. 2

6. 1/3

Practice Assignment – VII

1. (i) $\frac{n}{6}(4n^2 - 3n + 23)$ (ii) $\frac{n}{2}(n^3 + 4n^2 + 4n - 1)$ (iii) $\left[\frac{n(n+1)}{2}\right]^2 - \frac{3}{2}\left(3^n - 1\right)$

2. (i) $\frac{n}{3}(4n^2 + 6n - 1)$ (ii) $\frac{n^2(n+1)}{2}$ (iii) $\frac{1}{2}(3^n + 8n - 1)$

(iv) $\frac{n}{3}(n^2 + 8)$

3. (i) $\frac{n}{12}(n + 1)(3n^2 + 23n + 34)$ (ii) $\frac{n}{12}(n + 1)(9n^2 + 25n + 14)$

(iii) $\frac{n}{6}(n + 1)(n + 2)$ (iv) $3n(n + 1)(n + 3)$

(v) $\frac{n}{12}(n + 1)^2(n + 2)$ (vi) $\frac{n}{3}(4n^2 + 6n - 1)$

(vii) $\frac{1}{4}n(n + 1)(n + 4)(n + 5)$ (viii) $2n - 2 + \frac{1}{2^{n-1}}$

(ix) $\frac{3}{2}\left\{n - \frac{1}{2}\left(1 - \frac{1}{3^n}\right)\right\}$ (x) $\frac{1}{4}n(n + 3)$

(xi) $\frac{2n}{n+1}$ (xii) $\frac{n}{2(3n+2)}$

4. 8950
5. 10455
6. $\frac{n(n+1)(n+2)(3n+5)}{48}$
7. 42075
8. .....

Practice Assignment–VIII

1. 1/58
2. $2, \frac{6}{2n+1}$
3. (i) $\frac{25}{2}, \frac{100}{11}, \frac{50}{7}, \frac{100}{17}$ (ii) $\frac{6}{5}, \frac{3}{4}, \frac{6}{11}, \frac{3}{7}, \frac{6}{17}, \frac{3}{10}$
10. $16, 4; \frac{32}{5}$

Objective Assignment

1	D	11	B	21	A	31	C
2	C	12	C	22	D	32	D
3	B	13	A	23	A	33	B
4	A	14	A	24	A	34	C
5	C	15	C	25	A	35	D
6	C	16	C	26	C	36	B
7	A	17	A	27	A	37	C
8	A	18	C	28	B	38	C
9	A	19	C	29	C	39	A
10	D	20	A	30	B	40	C